AN

ELEMENTARY TREATISE

ON

ALGEBRA,

THEORETICAL AND PRACTICAL:

WITH AN APPENDIX ON

PROBABILITIES AND LIFE ANNUITIES.

BY

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FOURTH EDITION,

CONSIDERABLY ENLARGED.

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PREFACE

TO

THE SECOND EDITION.

THE first edition of the present Work was printed in octavo, and published at a price too high to warrant any very sanguine expectations as to the extent of its circulation. It gradually, however, found its way into the principal educational establishments of this kingdom—was adopted in the colleges of the United States—in the African College at the Cape of Good Hope—and in New South Wales.

This extensive patronage, unexpected alike by the publisher and myself, I attribute—not to any novelties contained in the book, but entirely to the efforts I had made to simplify as much as possible the more difficult parts of the subject, and thus to present to the young mathematical student a clear and perspicuous view of the fundamental principles of analytical calculation. Many complaints have, however, reached me from mathematical teachers, to the effect that the practical examples were not found sufficient in number fully to illustrate the theory. My own experience has proved to me the justness of this complaint, and has, moreover, led me to detect several other blemishes in the work.

In this new edition, it is hoped that these faults will be found in a great measure to have disappeared. The practical part has been considerably augmented throughout, the theory corrected and improved, and several new and interesting topics added. One subject, touched upon in the former edition, it has been thought advisable to exclude from this,—the chapter on the Theory of the Higher Equations; but the exclusion which has been thus made, as well as the additional matter which has been introduced, seemed equally necessary, to render the book better adapted to the wants of beginners, and more suitable for junior mathematical classes, in places of public education. Besides, in the progress of any science towards perfection, some departments of it are always found to acquire larger and more frequent accessions than others; these departments gradually increase in magnitude and importance, till at length, detaching themselves from the main body, they become objects of individual consequence and of distinct attention. Such has been the case with the Theory of Equations. That it is strictly a branch of pure Algebra there is no doubt; but it has exercised the talents and received the contributions of so many great men, and has, consequently, at length become of such interest and extent as to have assumed the form of a distinct department of analysis. The discussion of this subject is therefore consigned to a separate volume—*On the Analysis and Solution of Cubic and Biquadratic Equations;* which forms a supplement to the present work, and an introduction to the more extended treatise on *Equations in General.*

The following brief enumeration of the principal topics discussed in the book is extracted, with slight modifications, from the Preface to the former edition.

Chap. I contains the Preliminary Rules of the Science, in which the fundamental principles of operation are explained and illustrated.

Chap. II is on Simple Equations, and commences with some propositions preparatory to entering upon the solution of an equation, which operation they are intended to render more easy and inviting. Then follow the several methods of solving simple equations involving one, two, and three or more unknown quantities; each of these methods being illustrated separately, not only by algebraical examples, but also by practical questions; a mode rather different from that usually adopted, but which appears to be preferable, as it affords the student an early opportunity of applying the principles that he has acquired to useful and interesting inquiries, an exercise which is generally found to be peculiarly pleasing and encouraging.

Chap. III treats of Ratio, Proportion, and Progression, both arithmetical and geometrical; and, although the general formulas are fewer in number than those given in most books on this subject, yet it is shown that they are amply sufficient for every variety of case, and that therefore it would be superfluous to extend their number.

Chap. IV is on Irrational and Imaginary Quantities. The articles on irrational quantities have for their object the investigation of methods for reducing these quantities to the utmost simplicity of form, in order that the arithmetical computations implied in them may be rendered as little laborious as possible. The articles on Imaginary Quantities show how the ordinary operations of algebra, hitherto conducted with the symbols of real quantity, are to be carried on when, in certain hypotheses, those symbols cease to represent possible values, and indicate quantities purely imaginary.

Chap. V is on Quadratic Equations; and as this is a subject usually considered by students to be more difficult than either of those discussed in the preceding chapters, proportionate pains have been taken to render the modes of operation clear and intelligible; the solutions to some of the more complicated examples, which are given at length, will be of service to the student in cases of a similar nature; and will manifest to him how much a little judgment and ingenuity on his part will add to the simplicity and elegance of his operation.

Chap. VI contains the general investigation of the Binomial Theorem. The demonstration of this celebrated theorem in a manner adapted to elementary instruction, has always been considered as an object greatly to be desired, and many attempts have accordingly been made by different mathematicians for this purpose: all, however, that have yet appeared have been objected to, either on account of unwarrantable assumptions at the outset, which have consequently weakened the evidence, and rendered the demonstration incomplete, or because of a too tiresome and obscure method of reasoning, which has been incomprehensible to a learner. The demonstration given in this chapter is, I believe, different from any that has been previously offered, and appears to me to be more simple and satisfactory than any which I have had an opportunity of seeing. In the practical application of this theorem to the expansion of a binomial, it is always best to separate the case in which the exponent is *integral*, from that in which it is *fractional*, because, in the former instance, the process by the general formula is unnecessarily long and troublesome; a different method of proceeding is therefore usually pointed out; but it is rather singular that it has been ap-

plied only when the exponent is a *positive* integer: as, however, it is equally applicable when the exponent is a *negative* integer, it is here extended to that case.*

Chap. VII explains the nature and construction of Logarithms, and shows their importance in their application to several useful inquiries relating to interest, annuities, &c.

Chap. VIII is devoted to Series; and a new method for the summation of infinite series is given, which it is thought will be found to be more direct and easy than those generally used in elementary works. Several interesting subjects connected with series will be found in this chapter.

Chap. IX is on Indeterminate Equations of the first degree. In this chapter also some improvements will be found. The rule given at page 291 for solving an indeterminate equation involving two unknown quantities, is more direct and concise than the usual method, and equally simple.

Chap. X contains the principles of the Diophantine Analysis, or of indeterminate equations above the first degree, and concludes with a collection of diophantine questions; several of which are solved, in order to exhibit to the student the artifices which are sometimes to be employed in this part of the subject. This chapter has little claim to novelty, except as far as relates to the introduction of some new questions, and to the new solutions given to others.

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* In modifying the demonstration of the Binomial Theorem, for the present edition, the Author feels it incumbent upon him to state that he has consulted with advantage the demonstration given by Mr. Herapath, in the 'Philosophical Magazine' for May, 1819, as also that in Mr. Hind's 'Elements of Algebra.'

From the above outline an idea may be formed of the nature and pretensions of the work here submitted to the judgment of an impartial public; and if, upon examination, it shall be found that I have at all succeeded in my endeavours to lessen the labours of the student, it will afford me the highest satisfaction.

J. R. YOUNG.

Belfast College;
Aug. 19, 1834.

In this THIRD EDITION several changes, and, it is hoped, improvements have been introduced. The investigation of the Binomial Theorem has been entirely remodelled; and the several NOTES and SCHOLIA, interspersed throughout the work, have been so far amplified as to embrace many interesting inquiries connected with the theory of algebraic operations. The cautious student, who, as he goes along, feels his taste for his subject increase, will be glad to meet with these occasional digressions, more especially if he propose to advance in his mathematical studies beyond the bounds of elementary algebra. Readers of a different class may pass over this supplementary information without breaking the continuity of the developments and investigations whence the practical rules of operation are derived.

Belfast; May 25, 1839.

PREFACE

TO

THE FOURTH EDITION.

THE present FOURTH EDITION has been further amplified and improved in various parts. Some of the observations introduced will, I hope, tend to rectify the opinions sometimes advanced respecting certain alleged anomalies between arithmetic and algebra. These observations will be found in connexion with the discussion of the Binomial Theorem and the method of Indeterminate Coefficients, as also in a Note at the end of the book. The chapter on the Summation of Series has been extended, so as to include those classes of diverging series usually summed by the method of Increments, or by that of Finite Differences:—methods which the general principle I have employed in the chapter adverted to, seems entirely to supersede, as respects their application to the series there discussed. But the most important addition now made to the work is that which forms the APPENDIX, in which I have endeavoured to give a perspicuous though very brief sketch of the leading principles of the Theory of Probabilities:—a subject by far too much neglected in the ordinary courses of mathematical instruction.

In connexion with this subject of Probabilities, I have introduced some considerations having an immediate refer-

ence to Hume's celebrated argument against Miracles. I have not gone out of my way to discuss this topic. The value of human testimony is a legitimate object of investigation in the theory of probabilities. And I have thought it more interesting, as well as more instructive and important, to devote a small portion of the space at my disposal to an illustration of this kind, than to multiply examples drawn from games of chance:—from the turn up of a card or the cast of a die.

Eleven thousand seven hundred and fifty copies of this work have now been printed:—a fact which I may perhaps presume to consider as some proof that the book has answered its intended object; that object being to supply a full and perspicuous development of the principles of algebraic science, delivered in a form suited to the purposes of academical instruction, and, at the same time adapted to the exigencies of the unassisted student.

J. R. YOUNG.

Belfast; June, 1844.

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ERRATA.

Page 193, line 1, *for* (121) *read* (125).
.. .. 9 from bottom, *for* as *read* is.
249 .. 14, *for* 0 *read* 1.
264 .. 1, *for* $1 - x^3$ *read* $(1 - x)^3$.

AN

ELEMENTARY TREATISE ON ALGEBRA.

CHAPTER I.

PRELIMINARY RULES OF THE SCIENCE.

Definitions and Notation.

(*Article* 1.) ALGEBRA is that branch of Mathematics in which investigations are carried on, and the necessary calculations indicated by means of letters and signs: the letters being employed to represent the numbers or quantities entering into the inquiry, and the signs to indicate the operations to be performed on them. Although the letters thus employed to represent quantities are as much *signs* as the marks used to indicate the calculations to be performed on them, yet, for distinction, it is customary to call the former *symbols*, and to confine the term *signs* to the latter, including, however, under this designation, a few marks of occasional occurrence which are rather signs of *relation* than signs of operation. Thus the sign of equality, defined in Article (4), is in strictness a sign of relation; and the others here adverted to are enumerated and explained at the close of the definitions.

(2.) The sign $+$ (*plus*) denotes addition.
" $-$ (*minus*) denotes subtraction.

Thus, $a+b$ signifies that the quantity represented by b is to be *added* to that represented by a; and $a-b$ signifies that the quantity represented by b is to be *subtracted* from that represented by a. If, for instance, a represent 10, and b represent 2, then $a+b$ is 12, and $a-b$ is 8. If, however, it be intended to

express simply the *difference* between two quantities, without any reference as to which of them is the greater, then the sign of subtraction is modified into the wave mark $\sim$; so that $a \sim b$ means the *difference between a* and *b*.

(3.) $\times$ (*multiplied by*) denotes multiplication; and implies that the quantities between which it is placed are to be multiplied together. Thus, $a \times b$ signifies that the quantity represented by a is to be multiplied by that represented by b. In like manner $x \times y \times z$ implies that the three quantities or numbers for which x, y, and z stand are to be multiplied together. Multiplication is also denoted sometimes by a dot placed between the *factors*, or quantities multiplied, as $x.y.z$; but more frequently without any intervening sign between the factors; so that $x\,y\,z$ indicates the product of the three factors x, y, and z, as well as $x \times y \times z$ or $x.y.z$.

$\div$ (*divided by*) signifies that the former of the two quantities, between which it is placed, is to be divided by the latter; so that $a \div b$ implies that a is to be divided by b.

Division is also often denoted by placing the dividend over the divisor, and drawing a line between; thus, $\frac{a}{b}$ is the same as $a \div b$.

(4.) $=$ (*equal to*) denotes an equality of the quantities between which it is placed.

Thus, $a + b = 12$ signifies that a plus b is equal to 12; and $a - b = 8$ signifies that a minus b is equal to 8; also, $a + c - d = b + e$ denotes an equality between $a + c - d$ and $b + e$. This equality being of course understood to belong, not to the letters themselves, but to the things for which they stand. Any such assemblage of letters, however connected together, is called an *algebraical expression*.

(5.) A *power* of any quantity is the product of that quantity multiplied any number of times by itself. Thus, $a \times a$ is the second power or *square* of a, and is expressed in this manner, a^2; also, $a \times a \times a$ is the third power or *cube* of a, and is expressed thus, a^3; likewise a^4 is the fourth power of a; a^5 the fifth power of a, &c. The factors therefore which produce a *power* of any

quantity are each *equal* to that quantity; and, in the notation just exhibited, the small figure at the corner expresses the number of these equal factors in any proposed power of the quantity with which such small figure is connected.

(6.) A *root* of any quantity is a quantity which, if multiplied by itself a certain number of times, produces the original quantity; and it is called the *second* root, *third* root, &c., according to the number of multiplications. Thus the *second* or *square* root of a is a quantity whose *square* or *second power* produces a; the *third* or *cube root* of a is a quantity whose *cube* or *third power* produces a; the *fourth root* of a is a quantity whose *fourth power* produces a, &c. *Roots* are represented thus: $\sqrt[2]{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$, &c. or $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, &c., either of which forms respectively represent the *square root, cube root, fourth root*, &c. In the case of the *square root*, however, the 2 above the *radical sign* $\sqrt{}$ is usually omitted. Suppose $a=16$, then $\sqrt{a}$ or $a^{\frac{1}{2}}=4$, because 4×4, or $4^2 = 16$, also $\sqrt[4]{a}$, or $a^{\frac{1}{4}} = 2$, because $2 \times 2 \times 2 \times 2$, or $2^4=16$. The distinction therefore between power and root is sufficiently obvious: the power is produced by the multiplication of equal factors, the root is the factor itself, by the repetition of which the quantity proposed is produced.

(7.) If unity be divided by any *power* or *root* of a quantity, as for instance, by a^2, a^3, $a^{\frac{1}{2}}$ &c., the form of expression, agreeably to definition (p. 2), is $\frac{1}{a^2}$, $\frac{1}{a^3}$, $\frac{1}{a^{\frac{1}{2}}}$, &c. But this form being in many cases incommodious, the following notation is employed to represent the several *fractions*, viz. a^{-2}, a^{-3}, $a^{-\frac{1}{2}}$, &c. Unity divided by any quantity is called the *reciprocal* of that quantity; hence a^{-2} is the reciprocal of the square of a; a^{-3} is the reciprocal of the cube of a; $a^{-\frac{1}{2}}$ is the reciprocal of the square root of a, &c. The small figures used to denote powers and roots, as well as those employed, as above, to express the reciprocals of powers and roots, are called *indices* or *exponents*.

(8.) A *simple* quantity is that which consists of but one term, that is, a quantity not separated into parts by the intervention

of the sign plus or minus: such are the quantities a, ab, $4bc$, $6x^2y^3$, &c.

(9.) A *compound* quantity consists of two or more *simple* quantities, as $a + b$, $3ab - 2ad + e$, &c. If a compound quantity consist of but two terms, it is called a *binomial;* if of three terms, a *trinomial:* if of four terms, a *quadrinomial;* and if of more than four, a *polynomial* or *multinomial.*

(10.) The *coefficient* of a quantity is the number prefixed to it to denote how many times it is to be taken. Thus, $5x$ signifies five times the quantity x, and the number 5 is the *coefficient* of x. Also in the expressions $3xy$, $4abc$, $7yz$, &c., the coefficients are severally 3, 4, and 7; when no number is prefixed, as is the case when the quantity is to be taken but *once*, we say that the coefficient is unity; the quantities xy, abc, yz, &c. are in fact the same as $1xy$, $1abc$, $1yz$, the 1 being understood although it does not appear.

It is sometimes found convenient, when the coefficients are large numbers, to represent them, as well as the quantities which they multiply, by letters; choosing always, agreeably to Art. (1), the leading letters of the alphabet for this purpose; and hence arises the verbal distinction between *numeral* coefficients and *literal* coefficients. Thus, if we agree to represent the number 46852 by a, then $46852x$ may be more briefly written ax, where x has the literal coefficient a.

(11.) *Like* terms are those of which the *literal* parts, disregarding the coefficients, are the same; that is, however their coefficients may differ, the quantities to which they are prefixed are all alike. Thus the following terms with numeral coefficients are *like terms*, viz. $4ax$, $7ax$, $3ax$, ax, &c.; and so are these with literal coefficients; az, bz, cz, &c.

(12.) *Unlike* terms are those which consist of *different letters*, as the terms $4ab$, $7cd$, ef, &c.

(13.) A *vinculum* or bar ———, or a parenthesis ( ), is used to connect several quantities together. Thus, $\overline{a + x} \times b$, or, $(a + x)b$, signifies that the compound quantity $a + x$ is to be

multiplied by b; also, $\overline{2ac + 3b} \times \overline{4ax - 2by}$ signifies that $2ac + 3b$ is to be multiplied by $4ax - 2by$. The bar or vinculum is also sometimes placed vertically, thus:

$$\left.\begin{array}{r} a \\ -b \\ +c \end{array}\right| x \quad \text{is the same as } (a - b + c)\,x, \text{ or } \overline{a - b + c} \,.\, x.$$

(14.) The signs now explained comprehend all those which occur in the ordinary operations of common algebra. Of these the sign of equality, as already remarked in Article (1), denotes not operation but relation, viz., the relation of equality between two quantities, or between two *algebraical expressions.* The other signs of relation are the two *signs of inequality* $>$ and $<$, and the signs of proportionality, which are the dots interposed between the four terms of a proportion; and used both in arithmetic and algebra. The first of the above signs of inequality, viz. $>$, is placed between two quantities when we wish to express that the first of them is *greater* than the second; and the other sign is used when, on the contrary, the first is less than the second. Thus, to signify that a is *greater* than b, we write $a > b$; and to denote that a is *less* than b we write $a < b$.

To avoid the too frequent repetition of the word *therefore* or *consequently*, three dots, $\therefore$ are sometimes used; and these three dots inverted thus, $\because$ are employed by some algebraists for the word *because.*

NOTE. Quantities with the sign $+$ are called *positive* or *affirmative* quantities, or *additive* quantities; and those with the sign $-$ are called *negative* or *subtractive* quantities. A quantity to which no sign is prefixed is understood to be *positive*; we need, therefore, prefix the positive sign only as a means of connecting the quantity to which it belongs to one which precedes. Thus in the first example, page 6, the positive quantities $6ax$ and $7ax$ would be understood to be positive even if the $+$ before them were omitted; the insertion of this sign is therefore in these cases superfluous; but, in example 4, the positive quantity $4x$ requires the insertion of its sign to link it to the preceding quantity $6a$, which is itself positive, but has no such need of the sign.

ADDITION.

CASE I.

(15.) *When the quantities are like, but have unlike signs.*

1. Add the coefficients of all the *positive* quantities into one sum, and those of the *negative* quantities into another.

2. Subtract the *less* sum from the *greater.*

3. Prefix the sign of the *greater* sum to the remainder, and annex the common letters.

The reason of this is evident: for the value of any number of quantities, taken collectively, of which some are to be *added* and others to be *subtracted,* must be equal to the difference between all the *additive* quantities and all the *subtractive* quantities.*

EXAMPLES.

Add together the following quantities:

Ex. 1.	2.	3.
$+6ax$	$7xy$	$-4bx^2$
$-2ax$	$16xy$	$10bx^2$
$+7ax$	$-8xy$	$-7bx^2$
$-ax$	$2xy$	$-3bx^2$
Sum $10ax$	Sum $17xy$	Sum $-4bx^2$

* It thus appears that the operation called addition consists in determining the aggregate of a series of quantities of which some are subtractive and others additive. It is plain that as this aggregate refers simply to the *number* of quantities incorporated in the final amount, we have nothing to do with the *nature* of the quantities themselves; so that our ignorance of the actual things represented by the letters offers no obstacle to the finding of this numerical amount. Similar remarks obviously apply to subtraction, and to the other fundamental operations of algebra, these operations being independent of any particular signification given to the symbols to which they are applied.

4.

$$\begin{array}{r} 6a+4x \\ 4a+8x \\ -5a-2x \\ 7a-3x \\ \hline \text{Sum}\quad 12a+7x \\ \hline \end{array}$$

5.

$$\begin{array}{r} 2b+8x \\ -9b+7x \\ 4b+2x \\ 3b-4x \\ \hline \text{Sum}\quad 13x \\ \hline \end{array}$$

6.

$$\begin{array}{r} 4a^2+6bx \\ 3a^2+5bx \\ 7a^2-4bx \\ 2a^2+2bx \\ \hline \\ \hline \end{array}$$

7.

$$\begin{array}{r} 7\sqrt{y}-4(a+b) \\ 6\sqrt{y}+2(a+b) \\ 2\sqrt{y}+(a+b) \\ \sqrt{y}-3(a+b) \\ -9\sqrt{y}+5(a+b) \\ \hline \\ \hline \end{array}$$

8.

$$\begin{array}{r} 2ax^3-5(a+b)y^{\frac{1}{2}} \\ -7ax^3+2(a+b)y^{\frac{1}{2}} \\ -3ax^3+4(a+b)\sqrt{y} \\ 5ax^3-(a+b)\sqrt{y} \\ ax^3-3(a+b)y^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$$

9.

$$\begin{array}{r} 2(x^2+y^2)+7x^{\frac{1}{3}} \\ 5(x^2+y^2)-6x^{\frac{1}{3}} \\ -4(x^2+y^2)-2x^{\frac{1}{3}} \\ 16(x^2+y^2)-\sqrt[3]{x} \\ 3(x^2+y^2)+9x^{\frac{1}{3}} \\ \hline \\ \hline \end{array}$$

10.

$$\begin{array}{r} a(a+b)+3\sqrt{a-x} \\ -4a(a+b)+7\sqrt{a-x} \\ 11a(a+b)-6\sqrt{a-x} \\ -2a(a+b)-2\sqrt{a-x} \\ 5a(a+b)+14\sqrt{a-x} \\ \hline \\ \hline \end{array}$$

11.

$$\begin{array}{r} 7x^{\frac{1}{2}}y-2x\sqrt{y}+7 \\ x^{\frac{1}{2}}y+3xy^{\frac{1}{2}}+2 \\ 3x^{\frac{1}{2}}y-xy^{\frac{1}{2}}-6 \\ 9x^{\frac{1}{2}}y-4x\sqrt{y}-3 \\ -2x^{\frac{1}{2}}y+7x\sqrt{y}+1 \\ \hline \\ \hline \end{array}$$

12.

$$\begin{array}{r} 4(x+y)^{\frac{1}{2}}+\sqrt{xyz} \\ -7(x+y)^{\frac{1}{2}}+4\sqrt{xyz} \\ \sqrt{x+y}-3(xyz)^{\frac{1}{2}} \\ -3(x+y)^{\frac{1}{2}}+7(xyz)^{\frac{1}{2}} \\ -17(x+y)^{\frac{1}{2}}+2\sqrt{xyz} \\ -3\sqrt{x+y}+(xyz)^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$$

13.

$$\begin{array}{r} 5x\sqrt[3]{a+y}-2x\sqrt[4]{y}+\sqrt{2} \\ 3x(a+y)^{\frac{1}{3}}+6xy^{\frac{1}{4}}+2^{\frac{1}{2}} \\ -8x(a+y)^{\frac{1}{3}}-4xy^{\frac{1}{4}}+3\sqrt{2} \\ 7x\sqrt[3]{a+y}+3x\sqrt[4]{y}+2\sqrt{2} \\ 2x(a+y)^{\frac{1}{3}}+5x\sqrt[4]{y}+2^{\frac{1}{2}} \\ -9x\sqrt[3]{a+y}-8xy^{\frac{1}{4}}-8\sqrt{2} \\ \hline \\ \hline \end{array}$$

14.

$$
\begin{array}{rl}
-3\,(ax+by+cz)^{\frac{1}{4}} & -\sqrt{x^2+y^2}+a-b \\
2\sqrt[4]{ax+by+cz} & +(x^2+y^2)^{\frac{1}{2}}-3\,(a-b) \\
7\overline{ax+by+cz}^{\frac{1}{4}} & -\sqrt{x^2+y^2}+2\,(a-b) \\
3\sqrt[4]{(ax+by+cz)} & +(x^2+y^2)^{\frac{1}{2}}+a-b \\
-5\sqrt[4]{(ax+by+cz)} & +(x^2+y^2)^{\frac{1}{2}}-2\,\overline{a-b} \\
(ax+by+cz)^{\frac{1}{4}} & -\sqrt{x^2+y^2}-3\,\overline{a-b} \\
\hline
\end{array}
$$

CASE II.

(16.) *When both quantities and signs are unlike, or some like and others unlike.*

Find the value of the *like* quantities, as in the preceding case, and connect to this value, by their proper signs, the *unlike* quantities.

Thus, in the first of the following examples, we find that there are *four* quantities like $x^{\frac{1}{2}}$, viz. *two* in the *first* column, and *two* in the *second*,* whose value, by the former case, is $2x^{\frac{1}{2}}$; also, there are *three* quantities like ax, one in each column, whose value is $3ax$; there are, likewise, *three* quantities like ab, whose value is $-4ab$; and there are *four* quantities like xy, whose value is $4xy$; but, as x has no *like*, it is merely connected to the value of the *like* quantities by its *sign* $-$.

1.

$$
\begin{array}{l}
x^{\frac{1}{2}}+ax-ab \\
ab-\sqrt{x}+xy \\
ax+xy-4ab \\
x^{\frac{1}{2}}+\sqrt{x}-x \\
xy+xy+ax \\
\hline
\end{array}
$$

$$\text{Sum } 2x^{\frac{1}{2}}+3ax-4ab+4xy-x$$

* The student must not forget that $x^{\frac{1}{2}}$ and $\sqrt{x}$ are *like*, each expression representing the *square root* of x, (Def. 6, page 3.)

2.

$$\begin{array}{l}
3(c+d)x^3 + 4y - 2\sqrt{y} \\
6x^2y - 2ax + 12 \\
y + \sqrt{y} - 5ax \\
3ax - x^2y + (c+d)x^3 \\
x^2y + 2y - 14 \\
\hline
\text{Sum } 4(c+d)x^3 + 7y - \sqrt{y} + 6x^2y - 4ax - 2 \\
\hline
\end{array}$$

3.

$$\begin{array}{l}
\sqrt{x} + 3ax - 2\sqrt{b-x} \\
\frac{1}{2}ax - 2xy + 3\sqrt{x} \\
4xy + 3ax + (b-x)^{\frac{1}{2}} \\
\frac{1}{2}x^{\frac{1}{2}} + 8xy - 26 \\
17 - \sqrt{x} + ax \\
\hline
\\
\hline
\end{array}$$

4.

$$\begin{array}{l}
2ab + 12 - x^2y \\
x^{\frac{1}{2}}y + xy + 10 \\
3xy^{\frac{1}{2}} + 2x^2y - xy \\
5xy + 11 + x\sqrt{y} \\
17 - 2x^2y - x^{\frac{1}{2}}y \\
\hline
\\
\hline
\end{array}$$

5.

$$\begin{array}{l}
\sqrt{x^2+y^2} - \sqrt{x^2-y^2} + 2xy \\
(x^2-y^2)^{\frac{1}{2}} + \sqrt{x^2+y^2} - 3xy \\
4xy - (x^2-y^2)^{\frac{1}{2}} + \sqrt{x^2+y^2} \\
3(x^2+y^2)^{\frac{1}{2}} - 5xy + 7(x^2-y^2)^{\frac{1}{2}} \\
-7xy + 2\sqrt{x^2+y^2} - 3\sqrt{x^2-y^2} \\
\hline
\\
\hline
\end{array}$$

6.

$$\begin{array}{l}
2\sqrt[4]{xy} - 3a\sqrt{x} + x^2 - 13 \\
a\sqrt{x} + 12x^2 - 17 + (xy)^{\frac{1}{4}} \\
3x^2 - \sqrt{xy} + ax^{\frac{1}{2}} - 3 \\
-8(xy)^{\frac{1}{2}} + 9 - 2a\sqrt{x} + 3x^2 \\
x^2 + 3y^2 + 4\sqrt[4]{xy} - a\sqrt{x} \\
\hline
\\
\hline
\end{array}$$

(17.) When the coefficients are *literal* instead of *numeral*, they are to be collected in a similar way; but, as the numerical values of the coefficients are not in such cases actually exhibited, their aggregate amount can only be *indicated*, by connecting together the individual coefficients, with their proper signs, into one compound term. Thus, in the first example following, the literal coefficients a, cd, and b, when connected with their proper signs, plus, into one expression, furnish the compound coefficient $(a + cd + b.)$

1.

$$\begin{array}{r} ax + by^2 \\ cdx + ady^2 \\ bx - cy^2 \\ \hline (a + cd + b)\,x + (b + ad - c)\,y^2 \end{array}$$

2.

$$\begin{array}{r} ax + dy^2 \\ by - dx \\ -by^2 + my \\ \hline (a - d)\,x + (d - b)\,y^2 + (b + m)\,y \end{array}$$

3.

$$\begin{array}{l} pxy + mz^2 \\ qz + axy \\ nz^2 - bxy \\ 3xy + 2z \\ -z^2 - xy \\ \hline (a - b + p + 2)\,xy + (m + n - 1)\,z^2 + (q + 2)z \end{array}$$

4.

$$\begin{array}{r} abz^{\frac{1}{2}} + mn\sqrt{xy + 1} \\ cz^{\frac{1}{2}} + 4\,(xy + 1)^{\frac{1}{2}} \\ a\sqrt{xy + 1} - m\sqrt{z} \\ 2z + \tfrac{1}{2}z^{\frac{1}{2}} \\ 3z^{\frac{1}{2}} - 2z \\ \hline (ab + c - m + 3\tfrac{1}{2})\,z^{\frac{1}{2}} + (a + mn + 4)\,(xy + 1)^{\frac{1}{2}} \end{array}$$

5.

$$\begin{array}{l} x^2 + adz \\ \frac{1}{2}x^2 - nz \\ bz^2 + cez \\ dx^2 - mz \\ \hline \\ \hline \end{array}$$

6.

$$\begin{array}{l} \sqrt{x} + by \\ ax - z \\ amy + c\sqrt{x} \\ dz + y \\ \hline \\ \hline \end{array}$$

7.

$$\begin{array}{l} a\sqrt{x^2 - y^2} + b\sqrt{x^2 + y^2} \\ c\sqrt{x^2 + y^2} - d\sqrt{x^2 - y^2} \\ f(x^2 + y^2)^{\frac{1}{2}} - e(x^2 - y^2) \\ 2\sqrt{x^2 + y^2} + 4a\sqrt{x^2 - y^2} \\ \sqrt{x^2 - y^2} - (x^2 + y^2)^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$$

8.

$$\begin{array}{l} (a + b)\sqrt{x} - (2 + m)\sqrt{y} \\ 4y^{\frac{1}{2}} + (a + c)x^{\frac{1}{2}} \\ 3n\sqrt{y} - (2d - e)x^{\frac{1}{2}} \\ (m + n)y^{\frac{1}{2}} + (b + 2c)\sqrt{x} \\ -2n\sqrt{x} + 12a\sqrt{y} \\ \hline \\ \hline \end{array}$$

SUBTRACTION.

(18.) Place the quantities to be subtracted underneath those they are to be taken from, as in arithmetic. Then conceive the signs of the quantities in the *lower* line to be changed from + to —, and from — to +, and collect the quantities together as if it were addition.

For if a positive quantity, as b, is to be taken from another quantity, as a, the difference will be represented by $a - b$, which is obviously the same as the addition of a and $-b$; but if $b - c$ is to be *subtracted* from a, then, since b is greater than $b - c$ by c, if b be subtracted, too much will be taken away by c; consequently, c must be added to the remainder to make up the deficiency; therefore, the true remainder is equal to the addition of $-b$ and c; that is, it is equal, as in the former instance, to the *addition* of the quantities to be subtracted with their *signs changed*.

EXAMPLES.

1.

From $5xy + 2x^{\frac{1}{2}} - 7a$

Take $3xy - x^{\frac{1}{2}} + 2a$

Remainder $2xy + 3x^{\frac{1}{2}} - 9a$

2.

From $\sqrt{x+y} + 3ax - 12$

Take $4\sqrt{x+y} - 2ax + b$

Rem. $-3\sqrt{x+y} + 5ax - 12 - b$

3.

From $2(a-x)x^2 - 4(x+y) + z^2 + 2$

Take $(a-x)x^2 + 4z^2 - 2(x+y) - 6$

Rem. $(a-x)x^2 - 3z^2 - 2(x+y) + 8$

4.

From $3a(a-y) + 4by + a^3$

Take $2a(a-y) - 7by + 4a^3$

5.

$6x^2 + (x+y)^{\frac{1}{2}} - 10c$

$8x^2 - \sqrt{x+y} + 1$

6.

From $6abx + 12 - 3xy + 4xz$

Take $-3abx + xz - 7 + 5xy$

7.

From $\sqrt{x^2 - y^2} - 2(a+x)^{\frac{1}{2}} + 3$

Take $-3\sqrt{a+x} + 4(x^2 - y^2)^{\frac{1}{2}} - 1$

8.

From $2x(x+y)^{\frac{1}{4}} - 3axy + 2abc$

Take $-17axy + 11abc - x\sqrt[4]{x+y}$

Examples of Quantities with literal Coefficients.

1.

From $ax^2 + mxy + nx + b$
Take $sx^2 - pxy + qx - c$

$(a - s)x^2 + (m + p)xy + (n - q)x + b + c$

2.

From $a(x + y) - bxy + 4z^2$
Take $5(x + y) - cz^2 + (a + b)xy$

$(a - 5)(x + y) + (c + 4)z^2 - (a + 2b)xy$

The term $(a + b)xy$ in this example has a compound coefficient, viz. $(a + b)$, which coefficient, when its sign is changed, is $-(a + b)$, the minus implying that every term within the parenthesis is to be *subtracted* from what precedes. When, therefore, this compound coefficient, with its sign thus changed, is incorporated with $-b$, the coefficient of the like term in the upper row, another subtractive b becomes introduced, making the resulting coefficient $-(a + 2b.)$

3.

From $pxy + qxz - rz^2 + s$
Take $mxy - pqxz - nz^2 + a$

4.

From $a(x - y)^{\frac{1}{2}} + bxy + c(a + x)^2$
Take $(x - y)^{\frac{1}{2}} - bxy + (a + c)(a + x)^2$

5.

From $(a + b)(x + y) - (c + d)(x - y) + m$
Take $(a - b)(x + y) + (c - d)(x - y) - n$

6.

From $(a - b)xy - (p + q)\sqrt{x + y} - hx^2$
Take $(2p - 3q)(x + y)^{\frac{1}{2}} - axy - (3 + h)x^2$

MULTIPLICATION.

CASE I.

(19.) *When both multiplicand and multiplier are simple quantities.*

To the product of the coefficients annex the product of the letters, and it will be the whole product.

Thus, if it be required to multiply $6ax$ by $4b$, we have 24 for the product of the coefficients, and abx for the product of the letters; consequently, $24abx$ is the whole product; that is, $6ax \times 4b = 24abx$.*

NOTE. It must be particularly observed that quantities with *like* signs multiplied together, furnish a *positive* product whether the like signs be both $+$ or both $-$; and that quantities with *unlike* signs furnish a *negative* product. This may be expressed in short by the precept that, in multiplication, *like signs produce* plus, *and unlike signs* minus. The truth of this may be shown as follows:

1. Suppose any *positive* quantity, b, is to be multiplied by any other *positive* quantity, a; then b is to be taken as many times as there are units in a; and, as the sum of any number of positive quantities must be positive, the sign of the product ab must be $+$.

2. Suppose now that one factor b is negative, and the other a positive, then, as before, the product of $-b$ by a will be as many times $-b$ as there are units in a; and, since the sum of any number of negative quantities must be negative, the product in this case must be $-ab$.

3. If this last case be admitted, it will immediately follow that the product of $-b$ and $-a$ must be $+ab$; for if this be denied, the product must be $-ab$, so that $-b$ multiplied by $+a$ pro-

* When we speak of *multiplying* a quantity, we of course mean repeating that quantity a certain number of times: what that number of times is, in any proposed case, is expressed by the multiplier, and nothing more is expressed; so that the multiplier is necessarily a *number*, but the thing multiplied may be any quantity whatever.

duces the same as $-b$ multiplied by $-a$; which leads to the absurdity that $+a$ is the same thing as $-a$.*

NOTE. If *powers* of the same quantity are to be multiplied together, the operation is performed by simply *adding* the indices; thus, $a^2 \times a^3 = a^5$, for $a^2 = aa$, and $a^3 = aaa$, therefore $a^2 \times a^3 = aa \times aaa = aaaaa$, or a^5: also, $a^m \times a^n = a^{m+n}$, for $a^m = a \times a \times a \ldots$ to m factors, and $a^n = a \times a \times a \ldots$ to n factors, and therefore $a^m \times a^n = (aaa \ldots$ to m factors) $\times$ ($aaa \ldots$ to n factors), or (leaving out the sign $\times$) $= aaa \ldots$ to $m + n$ factors $= a^{m+n}$. From this it follows, that in the *division* of powers the indices are to be *subtracted*.†

---

* For a more detailed explanation of this *rule of signs*, the learner is referred to the *Catechism of Algebra*, which he may consult with advantage in reference to the first principles of algebraic calculation.

† This mode of proof does not apply when the quantities to be multiplied have fractional indices, although the rule still holds. Thus, let the product of $a^{\frac{1}{2}}$ and $a^{\frac{3}{5}}$ be required, then the exponents $\frac{1}{2}$, $\frac{3}{5}$, in a common denominator, are $\frac{5}{10}$, $\frac{6}{10}$; hence the proposed factors are the same as $a^{\frac{5}{10}}$ and $a^{\frac{6}{10}}$; that is, the 5th power of the 10th root of a, and the 6th power of the same root; we have, therefore, as above,

$$\left(a^{\frac{1}{10}}\right)^5 \times \left(a^{\frac{1}{10}}\right)^6 = \left(a^{\frac{1}{10}}\right)^{11} = a^{\frac{11}{10}} = a^{\frac{1}{2}+\frac{3}{5}}.$$

And similar reasoning will evidently apply to every other case.

That the number represented by $a^{\frac{5}{10}}$ is really the same as that represented by $a^{\frac{1}{2}}$ will appear, from considering that this number must be such that *ten* factors equal to it produce *five* a's, and consequently that *two* factors equal to it will produce *one* a; but the two factors which produce one a are represented each by $a^{\frac{1}{2}}$. Hence $a^{\frac{5}{10}}$ and $a^{\frac{1}{2}}$ represent the same thing. In like manner the number represented by $a^{\frac{6}{10}}$ is such that ten factors equal to it produce six a's, so that five such must produce three a's. But $a^{\frac{3}{5}}$ stands for the number whose fifth power is a^3; hence $a^{\frac{6}{10}}$ and $a^{\frac{3}{5}}$ represent the same number. The rule in the text is equally applicable when the exponents are negative. Thus the product of a^{-m} and a^{-n} is a^{-m-n}; for these factors are no other than $\frac{1}{a^m}$ and $\frac{1}{a^n}$ (Definition 7, p. 3), and from what is shown in the text,

EXAMPLES.*

1. Multiply $6\sqrt{ax}$ by $4b$.

$$\text{Here } 6\sqrt{ax} \times 4b = 24b\sqrt{ax}.$$

2. Multiply $3x^2y^2$ by $2ax$.

$$3x^2y^2 \times 2ax = 6ax^3y^2.$$

3. Multiply $12x^{\frac{1}{2}}y$ by $-4a$.
4. Multiply $-4x^3y^2$ by $-4x^2y^3$.
5. Multiply $6axy^2$ by $3a^2bx^2$.
6. Multiply $13a^3b^2xy^4$ by $-8abx^3y^2$.
7. Multiply $\frac{1}{2}x^2y^3z^4$ by $6x^4y^3z^2$.
8. Multiply $-9cxy^2z^5$ by $-\frac{1}{3}c^2x^2y^2z^2$.
9. Multiply $7a^2b^3x^4$ by $-\frac{1}{7}a^3b^2x$.
10. Multiply $5a^{\frac{1}{2}}b^{\frac{1}{2}}y^{\frac{1}{2}}$ by $3z^3$.
11. Multiply $\frac{4}{5}\sqrt{x}$ by $\frac{3}{4}abc$.
12. Multiply $-a^mx^n$ by $-a^nx^m$.

CASE II.

(20.) *When the multiplicand is a compound quantity, and the multiplier a simple quantity.*

Find the product of the multiplier and each term of the multiplicand *separately*, beginning at the left hand: connect these products by their proper signs, and the complete product will be exhibited.

---

and in the present note, the product of the denominators, whether the exponents be whole or fractional, is a^{m+n}, hence

$$\frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^{m+n}},$$

that is, by changing the notation,

$$a^{-m} \times a^{-n} = a^{-m-n} \text{ or } a^{-(m+n)}.$$

* Although the product of the letters will be the same in value in whatever order we arrange them, yet in these examples the student, conformably to the usual custom, is expected to arrange them according to their order in the alphabet.

EXAMPLES.

1. Multiply $ax + b$ by $4x^2$.

$$\begin{array}{l} ax \quad + b \\ 4x^2 \\ \hline 4ax^3 + 4bx^2 \\ \hline \end{array}$$

2. Multiply $12xy - ax + 6$ by $3xy$.

$$\begin{array}{l} 12xy - ax + 6 \\ 3xy \\ \hline 36x^2y^2 - 3ax^2y + 18xy \\ \hline \end{array}$$

3. Multiply $5ab + 3a - 2$ by $5xy$.
4. Multiply $12xy^2 - 4ax + a$ by $- 2ax$.
5. Multiply $12x^2y + 2xy^2 + xy$ by $3ax$.
6. Multiply $4abx + 3cy - abc$ by $3xy^2$.
7. Multiply $3x^2y^3 - 4xy^2 + bx$ by $- 7x^3y$.
8. Multiply $- 5axy^2 + \frac{1}{2}x^2 - \frac{1}{4}ay^3$ by $8axy$.
9. Multiply $\frac{1}{3}\sqrt{z} - \frac{2}{3}ax^2 - \frac{1}{2}xy^2$ by $- 6a^2x^2$.
10. Multiply $31xy^2 - 4\sqrt{x} + a$ by $- 2\sqrt{b}$.
11. Multiply $ax^2 - bx^3 + cx^4$ by $abcx$.
12. Multiply $a^mx^n + b^ny^m - c^2z^2$ by x^my^n.

CASE III.

(21.) *When both multiplicand and multiplier are compound quantities.*

Multiply each term in the multiplier by all the terms in the multiplicand.

Connect the several products by their proper signs, as in the last case, and their sum will be the whole product.

EXAMPLES.

1. Multiply $a + b$ by $a + b$.

$$\begin{array}{l}
a + b \\
\underline{a + b} \\
a^2 + ab \\
\qquad\;\; ab + b^2 \\
\hline
a^2 + 2ab + b^2 \\
\hline
\end{array}$$

2. Multiply $a + b$ by $a - b$.

$$\begin{array}{l}
a + b \\
\underline{a - b} \\
a^2 + ab \\
\quad\; - ab - b^2 \\
\hline
a^2 \qquad\;\; - b^2 \\
\hline
\end{array}$$

3. Multiply $2x^3 - 3x^2y + 4xy^2 - y^3$ by $2x^2 + 3xy - y^2$

$$\begin{array}{l}
2x^3 - 3x^2y + 4xy^2 - y^3 \\
\underline{2x^2 + 3xy - y^2} \\
4x^5 - 6x^4y + 8x^3y^2 - 2x^2y^3 \\
\qquad\quad 6x^4y - 9x^3y^2 + 12x^2y^3 - 3xy^4 \\
\qquad\qquad\qquad - 2x^3y^2 + 3x^2y^3 - 4xy^4 + y^5 \\
\hline
4x^5 \qquad\qquad - 3x^3y^2 + 13x^2y^3 - 7xy^4 + y^5 \\
\hline
\end{array}$$

4. Multiply $x + \frac{1}{2}y - 2$ by $\frac{1}{4}x + 3y$.

$$\begin{array}{l}
x + \frac{1}{2}y - 2 \\
\underline{\frac{1}{4}x + 3y} \\
\frac{1}{4}x^2 + \frac{1}{8}xy - \frac{1}{2}x \\
\qquad\quad 3xy + \frac{3}{2}y^2 - 6y \\
\hline
\frac{1}{4}x^2 + 3\frac{1}{8}xy - \frac{1}{2}x + \frac{3}{2}y^2 - 6y \\
\hline
\end{array}$$

5. Find the first four terms in the product of

$$a^m + a^{m-1}x + a^{m-2}x^2 + a^{m-3}x^3 + \&c. \text{ and } a + x.$$

$$\begin{array}{lllll}
a^m & + a^{m-1}x & + a^{m-2}x^2 & + a^{m-3}x^3 & + \&c. \\
a & + x & & & \\
\hline
a^{m+1} & + a^m x & + a^{m-1}x^2 & + a^{m-2}x^3 & + \&c. \\
& a^m x & + a^{m-1}x^2 & + a^{m-2}x^3 & + \&c. \\
\hline
a^{m+1} & + 2a^m x & + 2a^{m-1}x^2 & + 2a^{m-2}x^3 & + \&c.
\end{array}$$

6. Multiply $x - y$ by $x - y$.

Ans. $x^2 - 2xy + y^2$.

7. Multiply $x^2 + y^2$ by $x^2 - y^2$.

Ans. $x^4 - y^4$.

8. Multiply $4x^2 - 2xy + y^2$ by $2x + y$.

Ans. $8x^3 + y^3$.

9. Multiply $2x^3 - x^2y$ by $4x^2 - y^2$.

Ans. $8x^5 - 4x^4y - 2x^3y^2 + x^2y^3$.

10. What is the product of $4y^2 - 2yz + z^2$, $2y + z$ and $2y - z$?

Ans. $16y^4 - 8y^3z + 2yz^3 - z^4$.

11. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$.

Ans. $x^4 - y^4$.

12. Multiply $a^n + b^n$ by $a - b$.

Ans. $a^{n+1} + ab^n - a^nb - b^{n+1}$.

13. Multiply $a + x + x^2 + x^3 + x^4$ by $a - x$.

Ans. $a^2 + (a-1)x^2 + (a-1)x^3 + (a-1)x^4 - x^5$.

14. Multiply $x^4 - x^3 + x^2 - x + 1$ by $x^2 + x - 1$.

Ans. $x^6 - x^4 + x^3 - x^2 + 2x - 1$.

15. Multiply $3x^2 + (x+y)^{\frac{1}{2}} - 7$ by $2x^2 + \sqrt{x+y}$.

Ans. $6x^4 + (5x^2 - 7)\sqrt{x+y} - 14x^2 + x + y$.

16. Multiply $ax + bx^2 + cx^3$ by $1 + x + x^2 + x^3$.

Ans. $ax + \left.\begin{array}{l} a \\ b \end{array}\right| x^2 + \left.\begin{array}{l} a \\ b \\ c \end{array}\right| x^3 + \left.\begin{array}{l} a \\ b \\ c \end{array}\right| x^4 + \left.\begin{array}{l} b \\ c \end{array}\right| x^5 + cx^6$.

DIVISION.

CASE I.

(22.) *When both dividend and divisor are simple quantities.*

To the quotient of the *coefficients* annex the quotient of the *letters*,* and it will be the whole quotient.

NOTE. The rule for the signs must be observed here, as well as in multiplication.

EXAMPLES.

1. Divide $12ax$ by $3a$.

$$\frac{12ax}{3a}=4x.$$

2. Divide $24x^2y$ by $3xy$.

$$\frac{24x^2y}{3xy}=8x.$$

3. Divide $-16x^2y^2z^2$ by $-4xz$.

$$\frac{-16x^2y^2z^2}{-4xz}=4xy^2z.$$

4. Divide $9a^3x^4$ by $3ax^2$.
5. Divide $26ax^2y^2$ by $-2xy$.
6. Divide $-15b^3xy^5$ by $-bxy^2$.
7. Divide $28c^4z^6$ by $-7c^3z^6$.
8. Divide $-18a^2b^3y^7z^4$ by $-aby^3z$.
9. Divide $6a^{3n}x^{3m}y$ by $3a^nx^{2m}$.
10. Divide $-28x^5y^7z^n$ by $7y^7z^5$.
11. Divide $8ab^2x^3y^6$ by $2ax^2y^5$.
12. Divide $a^7b^8c^4x^2yz^9$ by $a^7bcx^2z^3$.

---

* The learner will readily discover the quotient of the *letters* by asking himself, *what letters must I join to those in the divisor to make those in the dividend?* Thus, in example 3, above, the dividend contains two x's, two y's, and two z's, while the divisor contains but one x and one z; so that to make up the letters in the dividend, I must join to those in the divisor one x, two y's, and one z; that is, the quotient of the letters will be xy^2z.

CASE II.

(23.) *When the dividend is a compound quantity, and the divisor a simple quantity.*

Find the quotient of the divisor and each term of the dividend *separately*, connect these quotients together by their proper signs, and the whole quotient will be exhibited.

EXAMPLES.

1. Divide $12a^2x + 4ax^2 - 16a$ by $4a$.

$$\frac{12a^2x + 4ax^2 - 16a}{4a} = 3ax + x^2 - 4.$$

2. Divide $a^{n+1}x - a^{n+2}x - a^{n+3}x - a^{n+4}x$ by a^n.

$$\frac{a^{n+1}x - a^{n+2}x - a^{n+3}x - a^{n+4}x}{a^n} = ax - a^2x - a^3x - a^4x.$$

3. Divide $8x^3y^2z^4 + 3xy^2z^3 - 2xy^3z^2 + x^2y^2z^2$ by xy^2z.

4. Divide $16a^4x^5 - 12a^5x^4 - 48a^6x^3 + 8a^7x^2$ by $-4a^4x^2$.

5. Divide $24a^2x^4 + 6a^3x^3 - 3a^4x^2 + 12ax$ by $-3ax$.

6. Divide $ax^n + ax^{n+1} + ax^{n+2} + ax^{n+3} +$ &c. by x^n.

7. Divide $6(x + y)^3 - 8(x + y)^2 + 4a^2(x + y)$ by $2(x + y)$.

8. Divide $ax^{m-1} + bx^{m+1} - cx^{m-3} + dx^5$ by x^{m-6}.

CASE III.

(24.) *When both dividend and divisor are compound quantities.*

1. Arrange both dividend and divisor according to the *powers of some letter* common to both; that is, let the *first* term, in both dividend and divisor, be that which contains the *highest power* of the same letter, the second the *next* highest, and so on.

2. Find how often the first term in the divisor is contained in that of the dividend, and it will give the first term in the quotient, by which all the terms in the divisor must be multiplied, and the product subtracted from the dividend.

3. To the remainder annex as many of the other terms of the dividend as are found requisite, and proceed to find the next term in the quotient, as in common arithmetic.

EXAMPLES.

1. Divide $x^6 - x^4 + x^3 - x^2 + 2x - 1$ by $x^2 + x - 1$.

$$
\begin{array}{l}
x^2 + x - 1)\ x^6 - x^4 + x^3 - x^2 + 2x - 1\ (x^4 - x^3 + x^2 - x + 1 \\
\qquad\qquad\ \ \underline{x^6 + x^5 - x^4} \\
\qquad\qquad\quad - x^5 + x^3 - x^2 \\
\qquad\qquad\quad \underline{- x^5 - x^4 + x^3} \\
\qquad\qquad\qquad\qquad x^4 - x^2 + 2x \\
\qquad\qquad\qquad\qquad \underline{x^4 + x^3 - x^2} \\
\qquad\qquad\qquad\qquad\quad - x^3 + 2x - 1 \\
\qquad\qquad\qquad\qquad\quad \underline{- x^3 - x^2 + x} \\
\qquad\qquad\qquad\qquad\qquad\quad x^2 + x - 1 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{x^2 + x - 1} \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{*\quad *\quad *}
\end{array}
$$

2. Divide $a^n - x^n$ by $a - x$.

$$
\begin{array}{l}
a - x)\ a^n - x^n\ (a^{n-1} + a^{n-2}x + a^{n-3}x^2 + a^{n-4}x^3 + \&c. \\
\qquad\ \ \underline{a^n - a^{n-1}x} \\
\qquad\qquad a^{n-1}x - x^n \\
\qquad\qquad \underline{a^{n-1}x - a^{n-2}x^2} \\
\qquad\qquad\qquad\quad a^{n-2}x^2 - x^n \\
\qquad\qquad\qquad\quad \underline{a^{n-2}x^2 - a^{n-3}x^3} \\
\qquad\qquad\qquad\qquad\qquad a^{n-3}x^3 - x^n \\
\qquad\qquad\qquad\qquad\qquad \underline{a^{n-3}x^3 - a^{n-4}x^4} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad a^{n-4}x^4 - x^n\ \&c.
\end{array}
$$

3. Divide $1+ax+bx^2+cx^3+dx^4+$ &c. by $1-x$.

$$1-x)\ 1+ax+bx^2+cx^3+dx^4\ \left(1+\left.\begin{array}{c}1\\a\end{array}\right|x+\left.\begin{array}{c}1\\a\\b\end{array}\right|x^2+\left.\begin{array}{c}1\\a\\b\\c\end{array}\right|x^3+\text{\&c.}\right.$$

$$\underline{1-x}$$

$$\left.\begin{array}{c}1\\a\end{array}\right|x+bx^2$$

$$\underline{\left.\begin{array}{c}1\\a\end{array}\right|x\ \left.\begin{array}{r}-1\\-a\end{array}\right|x^2}$$

$$\left.\begin{array}{c}1\\a\\b\end{array}\right|x^2+cx^3$$

$$\underline{\left.\begin{array}{c}1\\a\\b\end{array}\right|x^2\ \left.\begin{array}{r}-1\\-a\\-b\end{array}\right|x^3}$$

$$\left.\begin{array}{c}1\\a\\b\\c\end{array}\right|x^3+dx^4$$

$$\underline{\left.\begin{array}{c}1\\a\\b\\c\end{array}\right|x^3\ \left.\begin{array}{r}-1\\-a\\-b\\-c\end{array}\right|x^4}$$

&c. &c.

4. Divide a^5+x^5 by $a+x$.

Ans. $a^4-a^3x+a^2x^2-ax^3+x^4$.

5. Divide a^5-x^5 by $a-x$.

Ans. $a^4+a^3x+a^2x^2+ax^3+x^4$.

6. Divide $32x^5-16x^4y-8x^3y^2+4x^2y^3$ by $4x^2-y^2$.

Ans. $8x^3-4x^2y$.

7. Divide $16y^4-8y^3z+2yz^3-z^4$ by $4y^2-z^2$.

Ans. $4y^2-2yz+z^2$.

8. Divide $x^3+\frac{3}{4}x^2+\frac{3}{4}x+1$ by $\frac{1}{2}x+\frac{1}{2}$.

Ans. $2x^2-\frac{1}{2}x+2$.

9. Divide 1 by $1-x$.

Ans. $1+x+x^2+x^3+x^4+x^5+$ &c.

10. Divide $x^4 - y^4$ by $x^3 + x^2y + xy^2 + y^3$.

Ans. $x - y$.

11. Divide $y^{m+1} + yx^m - y^mx - x^{m+1}$ by $y^m + x^m$.

Ans. $y - x$.

12. Divide $a - bx + cx^2 - dx^3 +$ &c. by $1 + x$.

$$\text{Ans. } a \begin{array}{r|} -a \\ -b \end{array} \, x \begin{array}{r|} +a \\ +b \\ +c \end{array} \, x^2 - \text{\&c.}$$

SCHOLIUM.

From the preceding rules are deduced the following useful theorems, viz.

1. By the rule for addition, if the sum of any two quantities, a and b, be added to their difference, the amount will be twice the greater.*

2. By the rule for subtraction, if the difference of any two quantities be taken from their sum, the remainder will be twice the less.†

3. By multiplication, Article (21,) Example 2, page 18, if the sum of any two quantities be multiplied by their difference, the product will be the difference of their squares. It is of importance that these useful inferences be borne in mind.

ALGEBRAIC FRACTIONS.

The operations performed on Algebraic Fractions are similar to those performed on *numeral* fractions in Arithmetic: they are as follow:

To reduce a Mixed Quantity to an Improper Fraction.

(25.) Multiply the quantity to which the fraction is annexed by the denominator of the fraction; connect the product, by the

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* For $\left\{ \begin{array}{r} \ldots\ldots\ldots \; a + b \\ \text{added to } a - b \\ \hline \text{gives } 2a \end{array} \right.$ † and $\left\{ \begin{array}{r} \ldots\ldots\ldots\ldots\ldots \; a + b \\ \text{diminished by } a - b \\ \hline \text{gives} \quad 2b \end{array} \right.$

proper sign, to the numerator; place the denominator underneath, and we shall have the improper fraction required.

Thus, if it were required to reduce $ab - \frac{c}{d}$ to an improper fraction, then ab, the quantity to which the fraction is annexed, multiplied by d, the denominator, gives abd; which annexed to the numerator, c, by the proper sign $-$, gives $abd - c$; and this divided by d, gives $ab - \frac{c}{d}$, the mixed quantity proposed: therefore the equivalent improper fraction is $\frac{abd - c}{d}$.

EXAMPLES.

1. Reduce $(a + b) + \frac{ax}{y}$ to an improper fraction.

$$(a + b) + \frac{ax}{y} = \frac{(a + b)y + ax}{y}.$$

2. Reduce $3ax - \frac{a - b}{y}$ to an improper fraction.

$$3ax - \frac{a - b}{y} = \frac{3axy - (a - b)}{y}.$$

Here the expression $-(a - b)$ signifies that $a - b$ is to be *subtracted* from that which precedes, and therefore the signs of a and b must be *changed*, (Art. 18, page 11:) consequently,

$$\frac{3axy - (a - b)}{y} = \frac{3axy - a + b}{y};$$

the same must be observed in the following, and in every similar example.

3. Reduce $4x - \frac{3x - b + 4}{10}$ to an improper fraction.

$$4x - \frac{3x - b + 4}{10} = \frac{37x + b - 4}{10}.$$

4. Reduce $2ay + \frac{4x - ay + 2}{2xy}$ to an improper fraction.

Ans. $\frac{4axy^2 + 4x - ay + 2}{2xy}$.

5. Reduce $a-x-\frac{a^2-ax}{x}$ to an improper fraction.

Ans. $\frac{2ax-x^2-a^2}{x}$.

6. Reduce $ax-y-\frac{3ax-4y-2}{5}$ to an improper fraction.

Ans. $\frac{2ax-y+2}{5}$.

7. Reduce $a+b-\frac{a^2-b^2-3}{a-b}$ to an improper fraction.

Ans. $\frac{3}{a-b}$.

8. Reduce $x^4-x^3y+x^2y^2-xy^3+y^4-\frac{1}{x+y}$ to an improper fraction.

Ans. $\frac{x^5+y^5-1}{x+y}$.

To Reduce an Improper Fraction to a Whole or Mixed Quantity.

(26.) Divide the numerator by the denominator, and, if there be a remainder, place the denominator under it. Connect this fraction, by its proper sign, to the quotient, and we shall have the mixed quantity required.

Thus, if it were proposed to reduce $\frac{abd-c}{d}$ to a mixed quantity, we have only to perform the actual division of the numerator by the denominator, and we get for the *quotient* ab, and for the *remainder* $-c$; therefore $ab-\frac{c}{d}$ is the mixed quantity required.

EXAMPLES.

1. Reduce $\frac{4x^2+ax-2}{2x}$ to a mixed quantity.

$$\frac{4x^2+ax-2}{2x}=2x+\frac{ax-2}{2x}.$$

2. Reduce $\frac{2xy-a}{xy}$ to a mixed quantity. Ans. $2-\frac{a}{xy}$.

3. Reduce $\dfrac{x^2 - y^2 + 4}{x + y}$ to a mixed quantity. Ans. $x - y + \dfrac{4}{x+y}$.

4. Reduce $\dfrac{3(a^5 + b^5) - 3}{a + b}$ to a mixed quantity.

Ans. $3a^4 - 3a^3b + 3a^2b^2 - 3ab^3 + 3b^4 - \dfrac{3}{a+b}$.

5. Reduce $\dfrac{4axy^2 + 4x - ay + 2}{2xy}$ to a mixed quantity.

Ans. $2ay + \dfrac{4x - ay + 2}{2xy}$.

6. Reduce $\dfrac{x^3 - y^3 + x^2 - 2y^2}{x - y}$ to a mixed quantity.

Ans. $x^2 + xy + y^2 + x + y - \dfrac{y^2}{x - y}$.

To find the greatest Common Measure of the Terms of a Fraction.

(27.) Arrange the numerator and denominator according to the powers of some letter, as in division, making that the *dividend* which contains the *highest* power, and the other the divisor.

Perform the division, and consider the *remainder* as a new *divisor*, and the last divisor a new dividend; then consider the remainder that arises from this division as another new divisor, and the last divisor the corresponding dividend. Continue this process till the remainder is 0, and the last divisor will be the greatest common measure sought.

NOTE. If any factor be common to all the terms of either of the divisors, but *not* common to those of the corresponding dividend, this factor may be expunged from the divisor; or, if the terms of the dividend have a common factor which does not also enter into every term of the divisor, it may in like manner be expunged from the dividend; and hence a factor may always be *introduced*, without affecting the common measure.

The truth of the above process depends chiefly upon the two following properties:

Lemma 1. If a quantity divide another, it will also divide any

multiple of it: If, for instance, c divide b, and the quotient be n, it will also divide rb, and the quotient will be rn.

Lemma 2. If a quantity divide each of two others, it will also divide their sum and difference: For let c divide a, and call the quotient m; let it also divide b, and call the quotient n, then $a = mc$, and $b = nc$; therefore $a \pm b$* $= mc \pm nc$; now c evidently measures $mc \pm nc$, consequently it measures its equal, $a \pm b$.

(28.) Let now $\frac{a}{b}$ represent any fraction, and let the work in the margin be carried on according to the rule (Art. 27), c being put for the *first* remainder, d for the *second*, and so on, till the remainder at length becomes 0, and the work terminates. That the last divisor is the greatest common measure of a and b may be proved thus:

$$\begin{array}{l} b)\,a\,(r \\ \quad br \\ \quad \overline{} \\ \quad c)\,b\,(s \\ \qquad cs \\ \qquad \overline{} \\ \qquad d)\,c\,(t \\ \qquad\quad dt \\ \qquad\quad \overline{} \\ \qquad\quad e)\,d\,(u \\ \qquad\qquad eu \\ \qquad\qquad \overline{} \\ \qquad\qquad 0 \end{array}$$

Whatever divides a and b must divide a and br, (Lemma 1); and whatever divides these must divide their difference, c (Lemma 2): hence, whatever divides a and b divides also c. In like manner, whatever divides b and c must divide b and cs (Lemma 1); and whatever divides these must divide their difference, d (Lemma 2): hence, whatever divides a and b must also divide c and d. Similarly, whatever divides c and d must divide c and dt (Lemma 1), and whatever divides these must divide their difference, e (Lemma 2): hence, whatever divides a and b must also divide c, d, and e.

It follows therefore that every common measure of a and b is also a common measure of the series of divisors b, c, d, &c. Let e be the last divisor; then e measures d; therefore (Lemma 1) it measures dt; therefore (Lemma 2) it measures $e + dt$ or c; therefore it measures cs: and since it also measures d, it measures $d + cs$ or b; and thus, measuring b and c, it measures $c + br$ or a. It follows therefore that e measures a and b; and since it was

* The double sign $\pm$ signifies *plus* or *minus*.

before proved that whatever measures a and b must also measure e, it follows that e is the *greatest* common measure of a and b.*

With reference to the note, it may be observed, that, by expunging any factor common to all the terms of either of the quantities a, b, c, d, &c., but not common to those of the consecutive quantity, we do not interfere with any *common measure* which those quantities may have; neither should we do so, if, instead of a common factor being suppressed, one were to be introduced.

(29.) When the greatest common divisor of both numerator and denominator of a fraction is thus determined, the fraction may be simplified, or *reduced to its lowest terms.* For since the numerator has a divisor, as e, it is composed of two factors, as em; and since the denominator has also the same divisor, it likewise is formed from e, and another factor, as n; but, from the nature of division, it is evident that em divided by en will furnish the same quotient as m divided by n; in other words, $\frac{m}{n}$ is the same

* The method of finding the greatest common measure of any two quantities may be easily extended to finding the greatest common measure of three or more quantities. For let a, b, c, represent any three quantities, and let x be the greatest common measure of a and b, and y the greatest common measure of c and x; then, since whatever measures x measures also a and b; whatever measures c and x measures also a, b, c: therefore the greatest common measure of c and x is also the greatest common measure of a, b, and c; $\therefore y$ is the greatest common measure. If, again, z be the greatest common measure of y and d, then will z be also the greatest common measure of a, b, c, and d, &c. The chief use of the greatest common measure of the terms of a fraction is to reduce the fraction to its simplest form. In many fractions this common measure is discernible at sight; these, therefore, may be simplified without the aid of the above rule. But the process of finding the common measure is of importance on other accounts; as it is very remarkably connected with the important problem of the general solution of equations.

in value as $\frac{em}{en}$;* so that, by dividing the numerator and denominator by the common divisor e, we simplify the fraction without changing its value, and when this divisor is the greatest possible, the fraction is reduced to its simplest form. As we are thus at liberty to expunge any factor common to both numerator and denominator of a fraction, it follows that we may also introduce a common factor into both without disturbing the *value* of the fraction. The demonstrations of the rules for conducting the fundamental operations of fractions depend chiefly upon these obvious truths.

EXAMPLES.

1. Find the greatest common measure of the terms of the fraction

$$\frac{a^4 - x^4}{a^3 + a^2x - ax^2 - x^3}$$

Arranging the terms according to the powers of a,

$$a^3 + a^2x - ax^2 - x^3)\ a^4 - x^4\ (a - x$$

$$\begin{array}{r} a^4 + a^3x - a^2x^2 - ax^3 \\ \hline -a^3x + a^2x^2 + ax^3 - x^4 \\ -a^3x - a^2x^2 + ax^3 + x^4 \\ \hline 2a^2x^2 - 2x^4 \end{array}$$

$2a^2x^2 + 2x^4$

or *expunging* $2x^2$

$$a^2 - x^2)\ a^3 + a^2x - ax^2 - x^3\ (a + x$$

$$\begin{array}{r} a^3 - ax^2 \\ \hline a^2x - x^3 \\ a^2x - x^3 \\ \hline * \quad * \\ \hline \end{array}$$

---

* It is to be remembered that by the term *quotient* is to be understood that quantity, whether a number or a mere symbolical expression, which, when multiplied by the divisor, produces the dividend. If the actual division of m by n cannot be *performed*, it is agreed that the notation $\frac{m}{n}$ shall nevertheless indicate such division, and shall stand

Whence it appears that $a^2 - x^2$ is the greatest common measure of the terms of the proposed fraction; and, consequently, by dividing both numerator and denominator by this common measure, the fraction s reduced to its lowest terms, and becomes $\dfrac{a^2+x^2}{a+x}$.

2. It is required to reduce $\dfrac{a^4-x^4}{a^5-a^3x^2}$ to its lowest terms.

Arranging the terms according to the powers of a;

$$
\begin{array}{l|l}
a^5 - a^3x^2 & \\
\text{or} & a^4 - x^4 \; (a^2 + x^2 \\
a^2 - x^2 & a^4 - a^2x^2 \\
\hline
 & a^2x^2 - x^4 \\
 & a^2x^2 - x^4 \\
\hline
 & * \quad * \\
\hline
\end{array}
$$

The greatest common measure being $a^2 - x^2$, the fraction in its lowest terms is $\dfrac{a^2+x^2}{a^3}$.

3. Reduce the fraction $\dfrac{2ax^2 - a^2x - a^3}{2x^2 + 3ax + a^2}$ to its most simple form.*

Ans. $\dfrac{ax-a^2}{x+a}$.

4. Reduce the fraction $\dfrac{6ax^3 + ax^2 - 12ax}{6ax - 8a}$ to its lowest terms.

Ans. $\dfrac{2x^2+3x}{2}$.

5. Reduce the fraction $\dfrac{x^4-y^4}{x^3+y^3}$ to its lowest terms.

Ans. $\dfrac{x^3 - x^2y + xy^2 - y^3}{x^2 - xy + y^2}$.

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for the quotient; this symbol must therefore, when multiplied by the divisor n, produce the dividend m.

* In finding the common measure, either numerator or denominator may be taken as the first divisor, whichever is found the more convenient.

6. Reduce the fraction $\dfrac{3x^3 - 24x - 9}{2x^3 - 16x - 6}$ to its lowest terms. Ans. $\frac{3}{2}$.

(30.) From having the greatest common measure of two quantities, their least common multiple may be obtained, this being equal to the product of the two quantities divided by their greatest common measure: For, let x be the greatest common measure of a and b, and put $\frac{a}{x} = p$, and $\frac{b}{x} = q$; then p and q cannot have a common measure. Now, since $a = px$, and $b = qx$, and since pq is the least common multiple of p and q, and therefore pqx the least of px and qx; pqx (or its equal $\frac{ab}{x}$), must be the least common multiple of their equals, a and b. The least common multiple of three quantities is had by first finding that of two, and then the least common multiple of it and the other quantity, &c.

To Reduce Fractions to a Common Denominator.

(31.) 1. Multiply each numerator, separately, into all the denominators, except its own, and the products will be the new numerators.

2. Multiply all the denominators together, and the product will be the common denominator.

That this process alters the *form* merely, and not the *value* of the several fractions, will appear from observing that the numerator and denominator of each fraction are both multiplied by the same quantity, viz. by the product of the denominators of all the other fractions.

NOTE. If one of the given denominators should happen to be equal to the product of all the others, (as in Example 1, following,) then this denominator will obviously be the same as the common denominator, found by the above rule, for the other fractions; so that it will be sufficient to operate upon these others only, in order to reduce the whole to a common denominator.

The above rule will, of course, always effect the object desired: but *the least common multiple* of the original denominators will

furnish the *simplest* common denominator. Such common multiple is often readily discovered at sight, without applying the method at (30). The corresponding numerators will be obtained by multiplying each given numerator by the quantity which its denominator must be multiplied by, to produce the said common multiple.

EXAMPLES.

1. Reduce the fractions $\frac{a}{xy}$, $\frac{ax}{y}$, and $\frac{a}{x}$ to a common denominator.

$$\left.\begin{aligned} &axy \\ ax \times xy \times x &= ax^3y \\ a \times xy \times y &= axy^2 \end{aligned}\right\}\text{the new numerators.}$$

$$xy \times y \times x = x^2y^2 = \text{the common denominator;}$$

$\therefore$ the three fractions are $\frac{axy}{x^2y^2}$, $\frac{ax^3y}{x^2y^2}$, and $\frac{axy^2}{x^2y^2}$.

Since, however, in each of these fractions xy is common to both numerator and denominator, this quantity may be expunged, and the fractions written in the following more simple form:

$$\frac{a}{xy}, \frac{ax^2}{xy}, \frac{ay}{xy}$$

But, by attending to the above NOTE we arrive at once at these simplified forms. Thus, taking the second and third fractions only, as the product of their denominators gives the denominator of the first, the process will be

$$\left.\begin{aligned} ax \times x &= ax^2 \\ a \times y &= ay \end{aligned}\right\}\text{the new numerators;}$$

$$xy \text{ the common denominator;}$$

hence the three fractions are

$$\frac{a}{xy}, \frac{ax^2}{xy}, \frac{ay}{xy}.$$

2. Reduce the fractions $\frac{4}{ax}$, $\frac{2a}{x}$, and $\frac{3}{4}$, to a common denominator.

Here the least common multiple of the denominators is at once seen to be $4ax$: to produce this, the first denominator must be multiplied by 4, the second by $4x$, and the third by ax: hence, multiplying the numerators by these respectively, there results

$$\frac{16}{4ax}, \frac{8a^2}{4ax}, \text{ and } \frac{3ax}{4ax}.$$

3. Reduce $\frac{2x+1}{a}$ and $\frac{x+a}{3}$ to a common denominator.

Ans. $\frac{6x+3}{3a}$ and $\frac{ax+a^2}{3a}$.

4. Reduce $\frac{2x^2-a}{2a}$, a, and 4, to fractions having a common denominator.*

Ans. $\frac{2x^2-a}{2a}$, $\frac{2a^2}{2a}$, and $\frac{8a}{2a}$.

5. Reduce $\frac{3x^2-2}{4a}$, and $\frac{2x^2-x+4}{a+x}$ to a common denominator.

Ans. $\frac{3ax^2+3x^3-2x-2a}{4a^2+4ax}$ and $\frac{8ax^2-4ax+16a}{4a^2+4ax}$.

6. Reduce $\frac{a}{x+y}$, $\frac{b}{x-y}$, $\frac{c}{x^2-y^2}$, to a common denominator.†

Ans. $\frac{a(x-y)}{x^2-y^2}$, $\frac{b(x+y)}{x^2-y^2}$, $\frac{c}{x^2-y^2}$.

7. Reduce $\frac{a^2x}{1-x^2}$ and $\frac{a^2}{(1-x)^2}$ to a common denominator.

Ans. $\frac{a^2x(1-x)}{1-x-x^2+x^3}$ and $\frac{a^2(1+x)}{1-x-x^2+x^3}$.

8. Reduce $\frac{a-x}{x}$, $\frac{a+x}{x(a^2-x^2)}$, $\frac{a-x}{a+x}$, $\frac{1}{a-x}$, to a common denominator.

Ans. $\frac{(a+x)(a-x)^2}{x(a^2-x^2)}$, $\frac{a+x}{x(a^2-x^2)}$, $\frac{x(a-x)^2}{x(a^2-x^2)}$, $\frac{x(a+x)}{x(a^2-x^2)}$.

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* Whole quantities may be put under a fractional form by making their denominators unity: thus,

$$a=\frac{a}{1} \text{ and } 4=\frac{4}{1}, \text{ \&c.}$$

† See Note, page 32. The student will have frequent occasion for the property mentioned at page 24, viz., that *the sum multiplied by the difference of two quantities gives the difference of their squares, and the inference from it, that the difference of the squares is divisible by both the sum and difference of the quantities themselves.*

ADDITION OF FRACTIONS.

(32.) Reduce the fractions to a common denominator. Add the numerators together, and under the sum place the common denominator.

By thus connecting the numerators together into one compound quantity, and then writing the common denominator under it, we merely avoid the unnecessary repetition of that denominator; for the abridged form implies that every component part of the compound numerator is to be divided by the denominator, and this is all that is expressed by the several partial fractions, before the addition.

EXAMPLES.

1. Add together $\frac{2b}{x^2+b^2}$, and $\frac{1}{x}$.

$$\frac{2b}{x^2+b^2}+\frac{1}{x}=\frac{2bx}{x^3+b^2x}+\frac{x^2+b^2}{x^3+b^2x}=\frac{x^2+2bx+b^2}{x^3+b^2x}=\frac{(x+b)^2}{x(x^2+b^2)}$$

the sum required.

2. Add together $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$. Ans. $\frac{2x^2+2y^2}{x^2-y^2}$.

3. Add together $\frac{3x+2}{a}$, $\frac{4x+3}{b}$, and $\frac{5x+4}{c}$.

Ans. $\frac{bc(3x+2)+ac(4x+3)+ab(5x+4)}{abc}$.

4. Add together $\frac{2a}{b}$, $\frac{3a^2}{6}$, $\frac{2b}{a}$ and $\frac{1}{2}$.

Ans. $\frac{4a^2+a^3b+4b^2+ab}{2ab}$.

5. Required the sum of $\frac{x}{x^2-y^2}$, $\frac{y}{x+y}$, and $\frac{1}{x-y}$. (See NOTE, page 33.)

Ans. $\frac{2x+xy-y^2+y}{x^2-y^2}$.

6. Express $\frac{p}{3my^2-x}+\frac{y-6mpy^2}{(3my^2-x)^2}$ in a single fraction.

Ans. $\frac{y-3mpy^2-px}{(3my^2-x)^2}$.

SUBTRACTION OF FRACTIONS.

(33.) Reduce the fractions to a common denominator, and place this denominator under the difference of the numerators.

EXAMPLES.

1. Subtract $\frac{12x}{a}$ from $\frac{6ax}{5}$.

$\frac{6ax}{5} - \frac{12x}{a} = \frac{6a^2x}{5a} - \frac{60x}{5a} = \frac{(a^2-10)\,6x}{5a}$, the difference required.

2. Subtract $\frac{2x+1}{3}$ from $\frac{7x}{2}$. Ans. $\frac{17x-2}{6}$.

3. Subtract $\frac{3x+2}{x-1}$ from $\frac{5x-3}{x+1}$. Ans. $\frac{2x^2-13x+1}{x^2-1}$.

4. Subtract $\frac{1}{x+y}$ from $\frac{1}{x-y}$. Ans. $\frac{2y}{x^2-y^2}$.

5. Subtract $\frac{1}{x^2-y^2}$ from $\frac{1}{x-y}$. Ans. $\frac{x+y-1}{x^2-y^2}$.

6. Subtract $\frac{2x^2-13x+1}{x^2-1}$ from $\frac{5x-3}{x+1}$. Ans. $\frac{3x+2}{x-1}$.

7. Subtract $\frac{x-y}{x+y}$ from $\frac{2(x^2+y^2)}{x^2-y^2}$. Ans. $\frac{x+y}{x-y}$.

8. Subtract $\frac{x}{1-x^2}$ from $\frac{1}{(1-x)^2}$. Ans. $\frac{1+x^2}{(1+x)(1-x)^2}$.

MULTIPLICATION OF FRACTIONS.

(34.) 1. Multiply the numerators together, and it will give the numerator of the product.

2. Multiply the denominators together, and it will give the denominator of the product.

Let the fractions be $\frac{a}{b}$ and $\frac{c}{d}$; then we have to multiply the

quotient represented by $\frac{a}{b}$ by the quotient represented by $\frac{c}{d}$. From the nature of division the first quotient must be such, that when it is multiplied by the divisor b the product may be a; and the second quotient must be such, that when it is multiplied by the divisor d the product may be c. It follows, therefore, that the product of the quotients must be such, that when multiplied by the product of the divisors the final product may be ac. But $\frac{ac}{bd}$ is the quantity which, when multiplied by the product of the divisors, viz. by bd, produces ac. Hence $\frac{ac}{bd}$ is the product sought, and it is formed from the proposed fractions by multiplying as the rule directs.

EXAMPLES.

1. Multiply $\frac{4ax}{3}$ by $\frac{2a}{5}$.

Ans. $\frac{4ax}{3} \times \frac{2a}{5} = \frac{8a^2x}{15}$, the product required.

2. Multiply $\frac{2x+3y}{a}$ by $\frac{2a}{x}$.

Ans. $\frac{4x+6y}{x}$ = the product in its lowest terms.

3. Multiply $\frac{a-x^2}{2}$ by $\frac{2a}{a-x}$. Ans. $\frac{a^2-ax^2}{a-x}$.

4. Multiply $\frac{a+x}{a}$, $\frac{a-x}{x}$, and $\frac{a^2-x^2}{a^2+x^2}$ together.

Ans. $\frac{a^4-2a^2x^2+x^4}{a^3x+ax^3}$.

5. Multiply $\frac{x^2-y^2}{x}$, $\frac{x}{x+y}$, and $\frac{1}{x-y}$ together. Ans. 1.

6. Multiply $\frac{3(a^2-x^2)+a-x}{2}$ by $\frac{4}{3(a-x)}$.

Ans. $\frac{6(a+x)+2}{3}$.

DIVISION OF FRACTIONS.

(35.) Divide by the numerator of the divisor, and multiply by the denominator; or, which is the same thing, invert the divisor, and proceed as in multiplication.

For if $\frac{a}{b}$ is to be divided by $\frac{c}{d}$, the quotient must be such, that when it is multiplied by the divisor $\frac{c}{d}$ the product may be $\frac{d}{b}$. But $\frac{ad}{bc}$ is the quantity which, when multiplied by $\frac{c}{d}$, produces $\frac{a}{b}$. Hence $\frac{ad}{bc}$ is the quotient sought.

EXAMPLES.

1. Divide $\frac{4x+6}{3}$ by $\frac{x+3}{2x}$.

$$\frac{4x+6}{3} \times \frac{2x}{x+3} = \frac{8x^2+12x}{3x+9} = \text{the quotient.}$$

2. Divide $\frac{ax+b}{a}$ by $\frac{bx-a}{b}$. Ans. $\frac{abx+b^2}{abx-a^2}$.

3. Divide $\frac{6(a+x)+2}{3}$ by $\frac{4}{3(a-x)}$.

Ans. $\frac{3(a^2-x^2)+(a-x)}{2}$.

4. Divide $a+\frac{2ax-1}{b}$ by $\frac{x-a}{ax+1}$.

Ans. $\frac{a^2(bx+2x^2)+a(x+b)-1}{b(x-a)}$.

5. Divide 12 by $\frac{(a+x)^2}{x}-a$.

Ans. $\frac{12x}{a^2+ax+x^2}$.

6. Divide $\frac{a^4-2a^2x^2+x^4}{a^3x+ax^3}$ by $\frac{a^2-x^2}{a^2+x^2}$

Ans. $\frac{a^2-x^2}{ax}$.

INVOLUTION.

(36.) Involution is the raising of quantities to any proposed power.

If the quantity to be involved be a single letter, the involution is represented by placing the number of the power a little above it, as was observed in the definitions at the beginning.

The power of a simple quantity, consisting of more than one letter, is also similarly represented. Thus, the square or second power of abc is $(abc)^2$, or $\overline{abc}^2$, or $a^2b^2c^2$, the third power is $(abc)^3$, &c.

(37.) If the simple quantity be some power already, or if it be composed of factors that are powers, then the index, or indices, must be multiplied by the index of the power to which the quantity is to be raised. Thus, the second power of a^3 is a^6, because $a^3 \times a^3 = a^6$; also the nth power of a^3 is a^{3n}, because $a^3 \times a^3 \times a^3 \times a^3 \ldots\ldots$ to n factors is a^{3n}; the nth power of a^m is a^{mn}, because in like manner, $a^m \times a^m \times a^m \ldots\ldots$ to n factors is a^{mn}, whether m be whole or fractional. In the same manner the nth power of a^3b^2c is $a^{3n}b^{2n}c^n$, &c.

If the quantity have a coefficient, that coefficient must be raised to the proposed power, and prefixed to the power of the letters.

NOTE. If the quantity to be involved be negative, the signs of the *even* powers must be positive, and those of the *odd* powers negative, as is evident from the *rule of signs*, page 14.

EXAMPLES.

1. The square of $2ax$ is $4a^2x^2$.

2. The fourth power of $6a^2x$ is $1296a^8x^4$.

3. The third power of $-a^{\frac{1}{2}}b^{\frac{1}{3}}c$ is $-a^{\frac{3}{2}}bc^3$.*

---

* The quantity $a^{\frac{3}{2}}$ signifies the third power of $a^{\frac{1}{2}}$. The denominator of every such fractional exponent always expresses, agreeably to the notation explained in the definitions, the *root*, and the numerator the *power* of that root. If, for instance, a represented 4, then $a^{\frac{1}{2}}$ would represent its second or square root, viz. 2; and $a^{\frac{3}{2}}$ would express the cube or third power of this root, and would therefore signify 8.

4. The fourth power of $x^{\frac{1}{2}}y^{-\frac{1}{2}}$ is x^2y^{-2}.

5. The sixth power of $2\frac{a^2}{b}$ is $64\,\frac{a^{12}}{b^6}$.

6. The nth power of $3a^2x^3$ is $3^n a^{2n} x^{3n}$.

7. The fifth power of $a\sqrt[5]{xy}$ is a^5xy.

8. The fifth power of $\frac{x^{\frac{1}{2}}}{y^3}$ is

9. The seventh power of $-a^{-2}x^{-\frac{1}{2}}$ is

10. The fourth power of $-\frac{a^m}{2x^n}$ is

11. The nth power of $a^m x^m$ is

12. The nth power of $\frac{x^{\frac{1}{n}}}{y^{\frac{1}{n}}}$ is

(38.) When the quantity is compound, the involution is performed by actual multiplication.

EXAMPLES.

1. What is the fourth power of $a+b$?

$$
\begin{array}{rl}
 & a + b \\
 & a + b \\
\hline
 & a^2 + ab \\
 & \quad\;\; ab + b^2 \\
\hline
\text{The square} & a^2 + 2ab + b^2 \\
 & a + b \\
\hline
 & a^3 + 2a^2b + ab^2 \\
 & \qquad\; a^2b + 2ab^2 + b^3 \\
\hline
\text{The cube} & a^3 + 3a^2b + 3ab^2 + b^3 \\
 & a + b \\
\hline
 & a^4 + 3a^3b + 3a^2b^2 + ab^3 \\
 & \qquad\; a^3b + 3a^2b^2 + 3ab^3 + b^4 \\
\hline
\text{The fourth power} & a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
\hline
\end{array}
$$

2. What is the square of $a+b+c$?

$$\begin{array}{l}
a+b+c \\
a+b+c \\
\hline
a^2+ab+ac \\
\quad\quad ab+b^2+bc \\
\quad\quad\quad\quad ac+bc+c^2 \\
\hline
\end{array}$$

$$a^2+2ab+2ac+b^2+2bc+c^2=a^2+2ab+b^2+2c(a+b)+c^2$$
$$=(a+b)^2+2c(a+b)+c^2$$

In the involution of $a+b$ we observed that its square was equal to the square of a, + the square of b, + twice the product of a, b; and, in the square of $a+b+c$, by considering $a+b$ as one term, we have the same property, viz. it is equal to the square of $(a+b)$, + the square of c, + twice the product of $(a+b)$, c, as we have just seen; and it might also be shown, in a similar way, that the square of a quantity of four terms has the same property, by separating the first three terms, and considering them as a single term; and so on of any polynomial whatever.

3. Required the cube of $(a-x)$.

Ans. $a^3-3a^2x+3ax^2-x^3$.

4. Required the square of $4ax+x+1$.

Ans. $16a^2x^2+8ax^2+8ax+x^2+2x+1$.

5. Required the 4th power of $(a-x)$.

Ans. $a^4-4a^3x+6a^2x^2-4ax^3+x^4$.

6. Required the 4th power of $\sqrt{x^2+y^2}$.

Ans. $x^4+2x^2y^2+y^4$.

7. Required the 9th power of $\sqrt[3]{a+x}$.

Ans $a^3+3a^2x+3ax^2+x^3$.

8. Required the 5th power of $\sqrt[4]{x-y}$.

Ans. $(x-y)\sqrt[4]{x-y}$.

EVOLUTION.

(39.) Evolution is the extracting of roots.

The evolution of simple quantities is represented by indices, similarly to involution; and if the simple quantity have already an index, or if it be composed of factors having indices, the operation of evolution is performed by *dividing* the index or indices, being the reverse of the operation of involution: thus, the nth root of a^{mn} is a^m, since, by involution, the nth power of a^m is a^{mn}; the nth root of a^r is $a^{\frac{r}{n}}$, because the nth power of $a^{\frac{r}{n}}$ is a^r. In like manner, the nth root of $a^r b^s$ is $a^{\frac{r}{n}} b^{\frac{s}{n}}$, &c. And thus, from the known process of involution, and the previous employment of integral exponents to denote *powers*, fractional exponents naturally arise as the appropriate symbols of roots. The interpretation, therefore, of such exponents, would be immediately suggested, even if their meaning had not been explained in the chapter on Algebraical Notation.

It likewise appears that, since the division of the powers of the same quantity is performed by subtracting their indices, when the divisor is greater than the dividend, the quotient must be a quantity with a *negative* index: thus,

$$\frac{a^2}{a^3} = a^{2-3} = a^{-1}, \text{ that is } \frac{1}{a} = a^{-1}; \ \frac{a^2}{a^4} = a^{2-4} = a^{-2},$$

$$\text{that is } \frac{1}{a^2} = a^{-2}, \text{ \&c.}$$

And thus is suggested an interpretation of *negative* exponents, agreeing with the apparently arbitrary meaning already given to them in the chapter on Notation.

Again, since

$$\frac{a^n}{a^n} = a^{n-n} = a^0, \text{ and } \frac{a^n}{a^n} = 1,$$

it follows that a^0 is always the representative of 1, whatever be the value of a.

From what is said above, we see how the notation for powers

becomes extended to roots, and to the reciprocals of powers and roots,* all of which are appropriately represented by means of exponents or indices.

NOTE. Since the even powers of all quantities, whether positive or negative, are alike positive, (Art. 37, NOTE,) it follows that the even roots of all positive quantities may be either positive or negative; but the odd roots of a negative quantity must be negative, and, of a positive quantity, positive.

EXAMPLES.

1. The cube root of a^2x^6 is $a^{\frac{2}{3}}x^2$

2. The 5th root of $\dfrac{1}{a^2b^3}$ is $\dfrac{1}{a^{\frac{2}{5}}b^{\frac{3}{5}}}$ or $a^{-\frac{2}{5}}b^{-\frac{3}{5}}$.

3. The square root of $\dfrac{a^2x^3}{b^3c^4d^5}$ is $\dfrac{ax^{\frac{3}{2}}}{b^{\frac{3}{2}}c^2d^{\frac{5}{2}}}$, or $ax^{\frac{3}{2}}b^{-\frac{3}{2}}c^{-2}d^{-\frac{5}{2}}$.

4. The cube root of $-8a^{-3}b^6x^{-2}$ is $-2a^{-1}b^2x^{-\frac{2}{3}}$ or $-\dfrac{2b^2}{ax^{\frac{2}{3}}}$.

5. The 4th root of $\dfrac{16a^2b}{81c^2d^3}$ is $\dfrac{2a^{\frac{1}{2}}b^{\frac{1}{4}}}{3c^{\frac{1}{2}}d^{\frac{3}{4}}}$, or $2a^{\frac{1}{2}}b^{\frac{1}{4}} \times 3^{-1}c^{-\frac{1}{2}}d^{-\frac{3}{4}}$.

6. The square root of $\dfrac{4}{a^{\frac{1}{2}}b^3}$ is

7. The 4th root of $a^{-2}b^{-\frac{1}{2}}c$ is

8. The cube root of $-27a^7b^{\frac{1}{3}}x^{-8}$ is

9. The 5th root of $\dfrac{ab^{10}c^5}{d^2e^{\frac{1}{3}}m^2}$ is

10. The cube root of $\dfrac{a^{-1}}{b^nx^{\frac{1}{n}}}$ is

---

* The reciprocal of any quantity is unity divided by that quantity (Def. 8): thus, $\dfrac{1}{a^2}$ is the reciprocal of a^2, $\dfrac{1}{a+x}$ is the reciprocal of $a+x$, &c.

To Extract the Square Root of a Compound Quantity.

(40.) Arrange the terms according to the dimensions of some letter, and extract the root of the first term, which must always be a square: place this root in the quotient, subtract its square from the first term, and there will be no remainder.

Bring down the two next terms for a dividend; and put twice the root just found in the divisor's place, see how often this is contained in the first term of the dividend, and connect the quotient both to the last found root and to the divisor, which will now be completed. Multiply the complete divisor by the term last placed in the quotient, subtract the product from the dividend, and to the remainder connect the two next terms in the compound quantity, and proceed as before; and so on till all the terms are brought down.

The reason of the above method of proceeding will appear obvious from considering that, as the square of $a + b$ is $a^2 + 2ab + b^2$. (Art. 38), the square root of $a^2 + 2ab + b^2$ must be $a + b$. Now a is the root of the first term, whose square being subtracted, leaves $2ab + b^2$, the first term of which divided by $2a$ gives b, the other part of the root, which, connected to $2a$, completes the divisor $2a + b$, and this multiplied by b, the term last found, gives $2ab + b^2$ which finishes the operation; and these several steps agree with the rule.

$$
\begin{array}{r l}
 & a^2 + 2ab + b^2 \,(a + b \\
 & a^2 \\
\hline
2a + b) & 2ab + b^2 \\
 & 2ab + b^2 \\
\hline
 & * \quad\quad * \\
\hline
\end{array}
$$

If the root consist of three terms, $a + b + c$, its square will be $(a + b)^2 + 2c\,(a + b) + c^2$ (Art. 38); and we may return from this square to its root in a similar manner, viz. by finding first a, and then b, as above, and then deriving c from $(a + b)$ in the same way that b was derived from a; which is also according to the rule; and the same might be shown when the root consists of four, or a greater number of terms.

$$
\begin{array}{l|l}
\multicolumn{2}{l}{a^2 + 2ab + b^2 + 2c\,(a + b) + c^2\ [a + b + c} \\
\multicolumn{2}{l}{a^2} \\
\hline
2a + b & 2ab + b^2 \\
 & 2ab + b^2 \\
\hline
2\,(a + b) + c & 2c\,(a + b) + c^2 \\
 & 2c\,(a + b) + c^2 \\
\hline
 & * \quad * \quad * \\
\hline
\end{array}
$$

EXAMPLES.

1.

$$
\begin{array}{l|l}
\multicolumn{2}{l}{9x^4 - 12x^3 + 16x^2 - 8x + 4\ (3x^2 - 2x + 2} \\
\multicolumn{2}{l}{9x^4} \\
\hline
6x^2 - 2x & -12x^3 + 16x^2 \\
 & -12x^3 + 4x^2 \\
\hline
6x^2 - 4x + 2 & 12x^2 - 8x + 4 \\
 & 12x^2 - 8x + 4 \\
\hline
 & * \quad * \quad * \\
\hline
\end{array}
$$

2.

$$
\begin{array}{l|l}
\multicolumn{2}{l}{4x^6 + 12x^5 + 5x^4 - 2x^3 + 7x^2 - 2x + 1\ (2x^3 + 3x^2 - x + 1} \\
\multicolumn{2}{l}{4x^6} \\
\hline
4x^3 + 3x^2 & 12x^5 + 5x^4 \\
 & 12x^5 + 9x^4 \\
\hline
4x^3 + 6x^2 - x & -4x^4 - 2x^3 + 7x^2 \\
 & -4x^4 - 6x^3 + x^2 \\
\hline
4x^3 + 6x^2 - 2x + 1 & 4x^3 + 6x^2 - 2x + 1 \\
 & 4x^3 + 6x^2 - 2x + 1 \\
\hline
 & * \quad * \quad * \quad * \\
\hline
\end{array}
$$

3. Extract the square root of $4x^4 - 16x^3 + 24x^2 - 16x + 4$.

Ans. $2x^2 - 4x + 2$.

4. Extract the square root of $16x^4 + 24x^3 + 89x^2 + 60x + 100$.

Ans. $4x^2 + 3x + 10$.

5. Extract the square root of $1 + x$.

Ans. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}$, &c.

6. Extract the square root of $9x^6 - 12x^5 + 10x^4 - 28x^3 + 17x^2 - 8x + 16$.

Ans. $3x^3 - 2x^2 + x - 4$.

(41.) From the above method of extracting the square root of algebraical quantities, is derived that for the extraction of the square root of numbers, usually given in books on arithmetic. In order to extract the square root of a number, it is necessary first to ascertain how many figures the root sought will consist of; which may be done by placing a dot over every alternate figure, commencing with the units, the number of these dots will be the number of figures in the root; for since the square root of 100 is 10, the square root of every whole number less than 100 must consist of but one figure; and since the square root of 10000 is 100, it follows that the square root of every number between 100 and 10000 must consist of but two figures; likewise, since the square root of 1000000 is 1000, the square root of every number between 10000 and 1000000 must consist of but three figures, &c.: and in each of these cases the number of dots will be 1, 2, 3, &c. respectively.

Let it be required to find the square root of 56644.

Here the points indicate that the root consists of three figures, and the greatest square contained in 5 is 4; therefore the first figure of the root is 2, whose square taken from 5 leaves 1, to which the two next figures 66 being connected, gives 166 for the first dividend. Put 4, the double of the root figure, in the divisor's place, and it is contained in 16, 4 times; but, upon trial, 4 is found rather too great, the next figure of the root is therefore 3, which number, being placed in the divisor, completes it; and the product of

$$\begin{array}{r|l}
 & \dot{5}6\dot{6}4\dot{4}\ (238 \\
 & 4 \\
\hline
43 & 166 \\
 & 129 \\
\hline
468 & 3744 \\
 & 3744 \\
\hline
\end{array}$$

this divisor and last root figure taken from the dividend, leaves 37, to which the remaining two figures are connected, and the same operation repeated. To exhibit, however, more clearly the similarity between this and the algebraical process, let the figures of the number 56644 be represented according to their values in the arithmetical scale; thus, the value of the first figure 5 is 50000, that of the second 6000, of the third 600, of the fourth 40, and of the last 4. Now, as it is necessary that the first term should be a square, and as in this case it is not, it will be proper to substitute for 50000, 40000 + 10000, 40000 being the greatest square contained in it; the operation will then be as follows:

```
          40000 + 10000 + 6644 (200 + 30 + 8
          40000                      or
          -----                      238
  400 + 30 | 10000 + 6000
     or    |      or
    430    |    16000
   -----        12900
                -----
400 + 60 + 8 |  3100 + 600 + 40 + 4
     or      |         or
    468      |        3744
   -----              3744
                      ----
                      ----
```

To Extract the Cube Root of a Compound Quantity.

(42.) Arrange the terms according to the dimensions of some letter, and extract the root of the first term, which must be a cube; place this root in the quotient, subtract its cube from the first term, and there will be no remainder.

Bring down the three next terms for a dividend, and put three times the square of the root just found in the divisor's place, and see how often it is contained in the first term of the dividend, and the quotient is the next term of the root. Add three times the product of the two terms of the root, plus the square of the last

term, to the term already in the divisor's place, and the divisor will be completed.

Multiply the complete divisor by the last term of the root, subtract the product from the dividend, and to the remainder connect the three next terms, and proceed as before.

For (by Art. 38,) the cube of $a+b$ is

$$a^3+3a^2b+3ab^2+b^3;$$

and, from having the cube given, its root is found by the following process, being the same as that directed above, and which, after what has been said of the square root, does not seem to need any further explanation.

$$\begin{array}{r|l}
 & a^3+3a^2b+3ab^2+b^3\ (a+b \\
 & a^3 \\
\hline
3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\
\hline
 & * \quad * \quad * \\
\end{array}$$

If the root consist of three terms, a, b, c, they may be obtained by first finding a and b, as above, and then deriving c from $(a+b)$ in the same manner that b was derived from a.

It should be observed that the leading term in any divisor is all that is required to furnish the new term of the root, by help of which that divisor is to be completed; and this leading term is *the same for all the divisors.*

EXAMPLES.

1. Extract the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

$$\begin{array}{r|l}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1\ (x^2-2x+1 \\
 & x^6 \\
\hline
3x^4-6x^3+4x^2 & -6x^5+15x^4-20x^3 \\
 & -6x^5+12x^4-8x^3 \\
\hline
3x^4-12x^3+15x^2-6x+1 & 3x^4-12x^3+15x^2-6x+1 \\
 & 3x^4-12x^3+15x^2-6x+1 \\
\hline
 & * \quad * \quad * \quad * \quad * \\
\end{array}$$

2. Extract the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$.

$$\begin{array}{r|l}
 & x^6 + 6x^5 - 40x^3 + 96x - 64\ (x^2 + 2x - 4 \\
 & x^6 \\
\hline
3x^4 + 6x^3 + 4x^2 & 6x^5 - 40x^3 * \\
 & 6x^5 + 12x^4 + 8x^3 \\
\hline
3x^4 + 12x^3 - 24x + 16 & -12x^4 - 48x^3 + 96x - 64 \\
 & -12x^4 - 48x^3 + 96x - 64 \\
\hline
 & * \quad * \quad * \quad * \\
\end{array}$$

3. Extract the cube root of $8x^3 + 36x^2 + 54x + 27$.

Ans. $2x + 3$.

4. Extract the cube root of $27x^6 - 54x^5 + 63x^4 - 44x^3 + 21x^2 - 6x + 1$.

Ans. $3x^2 - 2x + 1$.

5. Extract the cube root of $a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

Ans. $a + b + c$.

From the foregoing method of extracting the cube root algebraically may be derived the numerical process for the cube root given in books of arithmetic. But this tedious operation is now entirely superseded by the easy and concise method which we have given in our chapter on Cubic Equations, contained in the *Treatise on the Analysis and Solution of Cubic and Biquadratic Equations*, which forms a supplement to the present volume †

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* In this example only two terms are brought down each time, instead of three, because in the proposed expression there are two terms absent, viz. those containing x^4 and x^2. If we write the expression thus,

$$x^6 + 6x^5 \pm 0x^4 - 40x^3 \pm 0x^2 + 96x - 64,$$

then three terms will, in effect, have been brought down, as in the preceding example, since $0x^4$ and $0x^2$ are each $= 0$.

† The author gladly avails himself of this opportunity to state, that he has recently found, in an early edition of *Ingram's Arithmetic*, a rule for the extraction of the cube root identical, in all its essential particulars, to that here referred to.

CHAPTER II.

ON SIMPLE EQUATIONS.

(43.) An Equation is an algebraical expression of equality between two quantities.

Thus, $4 + 8 = 12$ is an equation, since it expresses the equality between $4 + 8$ and 12; also, if there be an equality between $a - b + c$ and $f + g - h$, then $a - b + c = f + g - h$ expresses that equality, and is therefore an equation.

(44.) A Simple Equation, or an equation of the first degree, is that which contains the unknown quantity *simply*; that is, without any of its powers except the first.

(45.) A Quadratic Equation, or an equation of the second degree, is that which contains the square, but no higher power, of the unknown quantity.

(46.) An Equation of the third, fourth, &c. degree, is one in which the highest power of the unknown quantity is the third, fourth, &c. power.

(47.) And in general an Equation, in which the mth is the highest power of the unknown quantity, is called an equation of the mth degree.

Note. Each of the two members of an equation is called a side.

AXIOMS.

(48.) 1. If equal quantities be either increased or diminished by the same quantity, the results will be equal; or, in other words, if each side of an equation be either increased or diminished by the same quantity, the result will be an equation.

2. If each side of an equation be either multiplied or divided by the same quantity, the result will be an equation.

3. If each side of an equation be either involved to the same power, or evolved to the same root, the result will be an equation.

4. And generally, whatever operations be performed on one side of an equation, if the same operations be performed on the other side, the result will be an equation.

The three articles next following are devoted to exercises upon the preliminary operations usually involved in the solution of a simple equation. These are *transposition*, *clearing fractions*, and removing *radical signs*.

PROPOSITION.

(49.) Any term on one side of an equation may be transposed to the other side, provided its *sign* be changed.

For let $x + a - b = c + d$ be an equation, and add b to both sides; then (by axiom 1), $x + a - b + b = c + d + b$, that is, $x + a = c + d + b$, where b is transposed from the left to the right-hand side of the equation, and its sign changed. Again, subtract a from each side of this last equation, then $x + a - a = c + d + b - a$; that is, $x = c + d + b - a$, where a is transposed, and its sign changed; and in the same manner may any other term be transposed.

EXAMPLES.

Transpose all the terms containing the unknown quantity x, in the following equations, to the left-hand side, and the known terms to the right.

1. Given $4x + 12 = 2x - x + 21$.

Here $4x - 2x + x = 21 - 12$, the terms being transposed as required.

2. Given $\frac{x}{2} = 10 - \frac{x}{4} + \frac{x}{3}$.

Here $\frac{x}{2} + \frac{x}{4} - \frac{x}{3} = 10$.

3. Given $14 - x = 6x + 22$.

4. Given $\frac{4+x}{3} - x = \frac{6(x-2)}{5} - 8$.

5. Given $3x + 7 = 23 - 5x + \frac{4x-1}{2}$.

6. Given $ab + \frac{ax}{b} = a(x-b) + b$.

7. Given $5x + 8 - \frac{1}{2}x = 6 - \frac{2}{3}x + ax$.

8. Given $(2+x)(a-3) = 13 - 2ax$.

9. Given $(a+b)(c-x) = (x-a)b$.

PROBLEM I.

To clear an Equation of Fractions.

(50.) 1. Multiply each numerator by all the denominators, except its own, and the result will be an equation free from fractions; or,

2. Multiply every term by a *common multiple* of the denominators, and the denominators may then be expunged. If the *least* common multiple be used, the resulting equation will be in its lowest terms.*

The reason of the first of these methods is plain; for the multiplying the numerator of a fraction by its denominator is the same, in effect, as expunging the denominator; and multiplying every numerator by all the denominators, except its own, which is left out, or expunged, is the same as multiplying every term by the product of the denominators; each side of the equation is, therefore, multiplied by the same quantity, and therefore the results are equal (axiom 2).

The second method is equally obvious: for by each term being multiplied by a multiple of the denominators, the numerator of each fraction becomes divisible by its own denominator, and therefore this division may be actually performed.

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* If a common multiple be evident from inspection, this last method will generally be the best; but if not, the other method will be preferable.

EXAMPLES.

1. Clear the equation $\frac{1}{7}x + \frac{2}{3}x = 12 - \frac{3}{4}x$ of fractions.

By the first method the equation cleared of fractions is

$$12x + 56x = 1008 - 63x.$$

2. Clear the equation $\frac{x+6}{2} - 26 = \frac{5x}{4} + 2.$

Here 4 is evidently the least common multiple of the denominators;

$\therefore$ multiplying by 4,

$$2x + 12 - 104 = 5x + 8.$$

3. Clear the equation $\frac{4(x+3)}{5} - \frac{21}{4} = \frac{x}{6} - \frac{6x-8}{7} + 2.$

4. Clear the equation $\frac{2x+1}{3} - \frac{1}{7} = \frac{3x+5}{x-1}.$

5. Clear the equation $\frac{ax+b}{c} - \frac{a}{b} = \frac{cx+d}{ex}.$

6. Clear the equation $\frac{ax}{a+x} + b - \frac{a+x}{x} = 0.$

7. Clear the equation $\frac{x+3}{4} + 6 = \frac{2x-1}{3} + \frac{1}{2}.$

8. Clear the equation

$$\frac{4a(x+1)}{3} + \frac{2a(x-2)}{a} = \frac{a+x}{2a} + \frac{3}{5}.$$

9. Clear the equation

$$a + \frac{3a}{a+x} + 2 = \frac{4ax}{a-x} + \frac{x}{a^2-x^2}.$$

10. Clear the equation

$$\frac{ax}{a-x} + \frac{x}{a+x} = \frac{a}{a-x} + \frac{1}{a^2-x^2}.$$

11. Clear from fractions the equation

$$\frac{3-x}{2} + \frac{3}{5} = \frac{1}{20} + \frac{x-8}{10}.$$

12. Clear from fractions the equation

$$\frac{a+x}{\sqrt{a^2-x^2}} + \frac{\sqrt{a+x}}{\sqrt{a-x}} = \frac{\sqrt{a-x}}{\sqrt{a+x}}.$$

PROBLEM II.

To clear an Equation of Radical Signs.

(51.) 1. Transpose all the terms, except that under the radical, to one side of the equation.

2. Raise each side to the power denoted by the radical, and it will disappear. If there be more than one radical, this operation must be repeated.

EXAMPLES.

1. It is required to free from radicals the equation

$$\sqrt{a+x}+a=b.$$

By transposition, $\sqrt{a+x}=b-a$;

and by squaring each side, $a+x=b^2-2ab+a^2$.

2. It is required to free from radicals the equation

$$\sqrt{3x+\sqrt{x-6}}-2=3x.$$

By transposition, $\sqrt{3x+\sqrt{x-6}}=3x+2$;

and by squaring each side,

$$3x+\sqrt{x-6}=9x^2+12x+4;$$

or, by transposing,

$$\sqrt{x-6}=9x^2+12x+4-3x=9x^2+9x+4;$$

whence, by squaring again,

$$x-6=(9x^2+9x+4)^2=81x^4+162x^3+153x^2+72x+16.$$

3. It is required to free from radicals the equation

$$\sqrt[3]{a^2x+\sqrt{a^3x^3}}=b.$$

By cubing $a^2x+\sqrt{a^3x^3}=b^3$;

and by transposition, $\sqrt{a^3x^3}=b^3-a^2x$;

and by squaring, $a^3x^3=(b^3-a^2x)^2=b^6-2a^2b^3x+a^4x^2$.

4. It is required to free from radicals the equation

$$\sqrt{x+7} = \sqrt{x} + 1.$$

By squaring, in order to clear the first side

$$x + 7 = x + 2\sqrt{x} + 1;$$

and by transposing,

$$x + 7 - x - 1 = 2\sqrt{x}:$$

that is,

$$6 = 2\sqrt{x} \quad \therefore 3 = \sqrt{x},$$

and by squaring we have, finally,

$$9 = x.$$

5. Clear from radicals the equation

$$\sqrt{3 - x} + 6 = 8 + x.$$

6. Clear from radicals the equation

$$\sqrt{x} - 2 = 4 - 3\sqrt{x}.$$

7. It is required to free from radicals the equation

$$24 + \sqrt{ax + b} = 2x - a.$$

8. It is required to free from radicals the equation

$$a + \sqrt{-a + \sqrt{x+2}} = 3.$$

9. It is required to free from radicals the equation

$$\sqrt[3]{a + \sqrt{2ax}} = x.$$

10. It is required to free the equation

$$\sqrt{1 + \sqrt{x + \sqrt{ax}}} = 2 \text{ from radicals.}$$

11. It is required to free from radicals the equation

$$\sqrt{a-x} + 2 = 6 - \sqrt{x}.$$

12. It is required to free from radicals the equation

$$\sqrt[3]{x-4} - 1 = \sqrt[3]{2 + \sqrt{x}} - 1.$$

PROBLEM III.

To solve a Simple Equation containing but one Unknown Quantity.

(52.) 1. Clear the equation of fractions and radicals, if there be any.

2. Bring the unknown terms to one side of the equation, and the known terms to the other.

3. Collect each side into one term; and the unknown quantity, with a known coefficient, will form one side of the equation, and a known quantity the other side.

4. Divide each side by the coefficient of the unknown quantity, and the value of the unknown will be exhibited.

NOTE. Before performing any of the above operations, the equation may sometimes be previously simplified by the application of the 1st or 2d axiom, as will be seen in some of the following solutions.

EXAMPLES.

1. Given $4x+26=59-7x$, to find the value of x.

By transposition, $4x+7x=59-26$;

collecting the terms $11x=33$:

$\therefore$ dividing by 11, and we get $x=\frac{33}{11}=3$.

2. Given $\dfrac{x}{3}+6x=\dfrac{4x-2}{5}$, to find the value of x.

Clearing the equation $5x+90x=12x-6$;

and by transposition, $5x+90x-12x=-6$;

or collecting the terms $83x=-6$:

$\therefore$ dividing by 83, $x=-\frac{6}{83}$.

3. Given $\dfrac{3x+4}{5}-\dfrac{7x-3}{2}=\dfrac{x-16}{4}$, to find the value of x.

Here we immediately perceive that 20 is the least common multiple of the denominators:

$\therefore$ multiplying every term by 20,

$$12x+16-70x+30=5x-80\,;$$

and by transposition,

$$12x-70x-5x=-80-16-30\,;$$

or collecting the terms $-63x=-126$:

$\therefore$ dividing by -63, $x=\dfrac{-126}{-63}=2$.

4. Given $\dfrac{x+3}{7}-\frac{4}{5}=\dfrac{2(x-1)}{3}-\frac{9}{5}$, to find the value of x.

By transposition, $\dfrac{x+3}{7}-\dfrac{2(x-1)}{3}=\frac{4}{5}-\frac{9}{5}=-1$;

and clearing the equation, $3x+9-14x+14=-21$;

or by transposing, $3x-14x=-21-9-14$;

and collecting the terms $-11x=-44$;

$\therefore$ dividing by -11, $x=\dfrac{-44}{-11}=4$.

5. Given $\dfrac{6x-4}{3}-2=\dfrac{18-4x}{3}+x$, to find the value of x.

Multiplying every term by 3,

$$6x-4-6=18-4x+3x\,;$$

and by transposition, $6x+4x-3x=18+4+6$;

or collecting the terms $7x=28$:

$\therefore$ dividing by 7, $x=\frac{28}{7}=4$.

6. Given $\dfrac{3x+1}{3x}-\dfrac{3(x-1)}{3x+2}=\dfrac{9}{11x}$, to find the value of x.

Clearing the first side,

$$9x^2+9x+2-9x^2+9x=\frac{81x^2+54x}{11x}=\frac{81x+54}{11};$$

or collecting the terms on the first side,

$$18x+2=\frac{81x+54}{11};$$

and multiplying by 11, $198x + 22 = 81x + 54$;

or by transposing, $198x - 81x = 54 - 22$;

that is, $117x = 32$: $\therefore x = \frac{32}{117}$.

7. Given $\frac{x-2}{\sqrt{x}} = \frac{2\sqrt{x}}{3}$, to find the value of x.

Clearing the equation $3x - 6 = 2x$;

or by transposing, $3x - 2x = 6$; that is, $x = 6$.

8. Given $x + \sqrt{2ax + x^2} = a$, to find the value of x.

By transposition, $\sqrt{2ax + x^2} = a - x$;

and squaring each side, $2ax + x^2 = a^2 - 2ax + x^2$;

or by transposing, $2ax + 2ax = a^2 + x^2 - x^2$;

that is, $4ax = a^2$: and $\therefore x = \frac{a^2}{4a} = \frac{a}{4}$.

9. Given $2\sqrt{a^2 + x^2} = 4(a - \frac{1}{2}x)$, to find the value of x.

By squaring each side, $4a^2 + 4x^2 = 16a^2 - 16ax + 4x^2$;

and subtracting $4x^2$, $4a^2 = 16a^2 - 16ax$ (axiom 1);

or dividing by $4a$, $a = 4a - 4x$ (axiom 2;)

and by transposition, $4x = 4a - a$:

$$\text{and } \therefore x = \frac{3a}{4}.$$

10. Given $a + x = \sqrt{a^2 + x\sqrt{b^2 + x^2}}$, to find the value of x.

By squaring each side,

$$a^2 + 2ax + x^2 = a^2 + x\sqrt{b^2 + x^2};$$

and subtracting a^2, $2ax + x^2 = x\sqrt{b^2 + x^2}$ (axiom 1);

then dividing by x, $2a + x = \sqrt{b^2 + x^2}$ (axiom 2);

and squaring both sides, $4a^2 + 4ax + x^2 = b^2 + x^2$

or subtracting x^2, $4a^2 + 4ax = b^2$ (axiom 1);

and by transposition, $4ax = b^2 - 4a^2$:

$$\therefore x = \frac{b^2 - 4a^2}{4a}.$$

11. Given $\frac{3x}{2} + 7 = 17 - x$, to find the value of x. Ans. $x = 4$.

12. Given $\frac{5x+2}{3} - 8 = 2x - 11$, to find the value of x.

Ans. $x = 11$.

13. Given $\frac{4(x+2)}{3} - 1 = \frac{3x+1}{2}$, to find the value of x.

Ans. $x = 7$.

14. Given $\frac{x-1}{7} + \frac{x+4}{3} = x - 3$, to find the value of x.

Ans. $x = 8$.

15. Given $\frac{x}{2} - \frac{x}{3} + 5 = \frac{3(x+2)}{4}$, to find the value of x.

Ans. $x = 6$.

16. Given $\frac{312+5x}{9} + 3 = \frac{4(78-x)}{9}$, to find the value of x.

Ans. $x = -3$.

17. Given $\frac{x+40}{5} + \frac{x}{3} - 8 = 47 - \frac{x}{4}$, to find the value of x.

Ans. $x = 60$.

18. Given $\frac{5(x+3)}{7} + 1 = \frac{10(x-1)}{3} - 4$, to find the value of x.

Ans. $x = 4$.

19. Given $\frac{2}{x+2} + \frac{x}{4} = \frac{x^2+1}{4x}$, to find the value of x.

Ans. $x = \frac{2}{7}$

20. Given $\frac{1}{a^2-x^2} - a = \frac{ax}{a-x} + \frac{a}{a+x}$, to find the value of x.

Ans. $x = \frac{a^2+a^3-1}{a-a^3}$.

21. Given $\frac{(a-b)x}{2} + \frac{x}{3} = \frac{ab}{4} + a$, to find the value of x.

Ans. $x = \frac{3a(b+4)}{6(a-b)+4}$.

22. Given $\frac{2}{3}x^2 + \frac{1}{2}x = x + \frac{x^2+x}{4}$, to find the value of x.

Ans. $x = 1\frac{4}{5}$.

23. Given $4abx^2 = \frac{3ax^2 - 2bx + ax}{3}$, to find the value of x.

Ans. $x = \frac{a - 2b}{12ab - 3a}$.

24. Given $21 + \frac{3x - 11}{16} = \frac{5(x-1)}{8} + \frac{97 - 7x}{2}$, to find the value of x.

Ans. $x = 9$.

25. Given $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{5} = 77$, to find the value of x.

Ans. $x = 60$.

26. Given $x + \frac{a}{b}x + \frac{c}{b}x = m$, to find the value of x.

Ans. $x = \frac{bm}{a + b + c}$.

27. Given $\frac{a^2 + x^2}{ax} = b + \frac{x}{a}$, to find the value of x.

Ans. $x = \frac{a}{b}$.

28. Given $\frac{c}{a + bx} - \frac{d}{e + fx} = 0$, to find the value of x.

Ans. $x = \frac{ad - ce}{cf - bd}$.

29. Given $(a + x)(b + x) - a(b + c) = \frac{a^2c}{b} + x^2$, to find the value of x.

Ans. $x = \frac{ac}{b}$.

30. Given $\sqrt{3x - 1} = 2$, to find the value of x. Ans. $x = \frac{5}{3}$.

31. Given $\sqrt{x + x^2} = x + \frac{1}{3}$, to find the value of x. Ans. $x = \frac{1}{3}$.

32. Given $3\sqrt{2x + 6} + 3 = 15$, to find the value of x.

Ans. $x = 5$.

33. Given $\sqrt[3]{3x + 13} - 4 = 0$, to find the value of x.

Ans. $x = 17$.

34. Given $\sqrt{x} + 3 = \sqrt{21 + x}$, to find the value of x.

Ans. $x = 4$.

35. Given $\frac{\sqrt{a^2 - y^2}}{\sqrt{a - y}} + y = a + 2y$, to find the value of y.

Ans. $y = 1 - a$.

36. Given $x+\sqrt{a-x}=\dfrac{a}{\sqrt{a-x}}$, to find the value of x.

Ans. $x=a-1$.

37. Given $\sqrt{4+\sqrt{x^4-x^2}}=x-2$, to find the value of x.

Ans. $x=2\frac{1}{8}$.

38. Given $(2+x)^{\frac{1}{2}}+x^{\frac{1}{2}}=4(2+x)^{-\frac{1}{2}}$, to find the value of x.

Ans. $x=\frac{2}{3}$.

39. Given $x^2+x^{-2}=(x-x^{-1})^2+x$, to find the value of x.

Ans. $x=2$.

40. Given $\dfrac{31+4x}{3}-\dfrac{3x+47}{8}-\dfrac{3x-19}{16}=47\frac{3}{4}+\dfrac{16-10x}{11}-\dfrac{5x+20}{7}$, to find the value of x.

Ans. $x=17$.

QUESTIONS PRODUCING SIMPLE EQUATIONS INVOLVING BUT ONE UNKNOWN QUANTITY.

(53.) In order to resolve a question algebraically, the first thing to be done is to consider attentively its conditions; then, having represented the quantity or quantities sought, by x, or x, y, &c. if we perform with it, or them, and the known quantities, the operations that are described in the question, we shall finally obtain an equation from which the values of the assumed letters x, y, &c. may be determined.

Instead of representing the unknown quantity by x or y, &c. it will sometimes be found more convenient to represent it by $2x$ or $2y$, or by $3x$, $3y$, &c. for the purpose of avoiding the introduction of fractional expressions in those cases where a half, a third, &c. of the unknown quantity is directed to be taken; (see Question VII, following.) When we see by the conditions of the question that several different fractional parts of the unknown quantity will occur in the algebraical statement of those conditions, it will be advisable to represent the unknown by such a multiple of x or of y as will be actually divisible into the proposed parts. (See Question II following.)

QUESTION I.

It is required to find a number, such that, if it be multiplied by 4 and the product increased by 3, the result shall be the same as if it were increased by 4 and the sum multiplied by 3.

Let x represent the number sought; then, if it be multiplied by 4, and the product increased by 3, there will result $4x+3$; but this result, according to the question, must be the same as $x+4$ multiplied by 3; hence we have this equation, viz.

$$4x+3=3x+12;$$

and by transposition, $4x-3x=12-3$;

that is, $x=9$, the number required.

QUESTION II.

It is required to find a number, such that its third part increased by its fourth part shall be equal to the number itself diminished by 10.

Let x represent the number.

Then, by the question, $\frac{x}{3}+\frac{x}{4}=x-10$;

or clearing the equation, $4x+3x=12x-120$;

and by transposition, $4x+3x-12x=-120$;

that is, $-5x=-120$:

$$\therefore x=\frac{-120}{-5}=24, \text{ the number required.}$$

We might have avoided fractions in the statement of the conditions of this question, by representing the number sought not by x, but agreeably to the directions above, by such a multiple of x as would really divide by 3 and 4. Choosing the least multiple, the process will be as follows:

Let $12x$ be the number.

Then, by the question, $4x + 3x = 12x - 10$;

and by transposition, $4x + 3x - 12x = -10$;

that is, $-5x = -10$ $\therefore x = \frac{-10}{-5} = 2$:

$\therefore 12x = 24$, the number required.

QUESTION III.

A person left 350*l.* to be divided among his three servants, in such a way that the first was to receive double of what the second received, and the second double of what the third received. What was each person's share?

Let the share of the third be represented by x;

then that of the second was . . . . . $2x$;

and that of the first . . . . $4x$;

and, since the sum of their shares amounts to 350*l.*,

we have $x + 2x + 4x = 350$;

or $7x = 350$;

and $\therefore x = \frac{350}{7} = 50$;

whence the share of the third was . . £50

of the second . . . 100

of the first . . . 200

QUESTION IV.

It is required to divide 160*l.* among three persons, in such a manner that the first may receive 10*l.* more than the second, and the second 12*l.* more than the third.

Let the share of the third be x;

then that of the second is . $x + 12$;

and that of the first . . $x + 12 + 10$;

and by the question, $x + x + 12 + x + 12 + 10 = 160l.$;

that is, by addition, and transposition, $3x = 126$;

$$\text{whence, } x = \tfrac{126}{3} = 42;$$

$\therefore$ the share of the third is . . £42

second . . 54

first . . . 64.

QUESTION V.

A merchant has spirits at 9 shillings and at 13 shillings per gallon, and he wishes to make a mixture of 100 gallons that shall be worth 12 shillings per gallon. How many gallons of each must he take?

Suppose x to be the number of gallons at 9*s.*;

then $100 - x$ must be the number at 13*s.*;

also the value of the x gallons, at 9*s.*, is $9x$ shillings;

and of the $100 - x$, at 13*s.*, is $1300 - 13x$ shillings;

and the value of the whole mixture, at 12*s.*, is 1200*s.*:

$$\therefore 9x + 1300 - 13x = 1200;$$

$$\text{that is, } -4x = 1200 - 1300 = -100;$$

$$\text{consequently, } x = \frac{-100}{-4} = 25:$$

$\therefore$ there must be 25 gallons at 9*s.*

and $100 - 25 = 75$ . . 13*s.*

QUESTION VI.

How many gallons of spirits, at 9*s.* a gallon, must be mixed with 20 gallons at 13*s.*, in order that the mixture may be worth 10*s.* a gallon?

Let x be the number of gallons at 9*s.*, the value of which will be $9x$ shillings; also $x + 20$ will be the whole number of gallons in the mixture, the value of which, at 10*s.*, is $10x + 200$ shillings; now the value of the 20 gallons at 13*s.* is 260 shillings:

$$\therefore 9x + 260 = 10x + 200;$$

$$\text{and by transposition, } 260 - 200 = 10x - 9x;$$
$$\text{that is, } 60 = x:$$

$\therefore$ there must be 60 gallons at 9*s*., in order that the mixture, which will contain 80 gallons, may be worth 10*s*. a gallon.

QUESTION VII.

A fish was caught whose tail weighed 9 lbs.; his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail together. What was the weight of the fish?

Let $2x$ be the number of lbs. the body weighed;

then $9 + x =$ weight of the head;

and, since the body weighed as much as both head and tail, we have

$$2x = 9 + 9 + x;$$

or by transposition, $2x - x = 9 + 9$;

$$\text{that is, } x = 18:$$

$\therefore$ 36 lbs. = weight of the body,

$9 + x = 27$ lbs. = ... head,

9 lbs. = ... tail.

The sum = 72 lbs. the whole weight of the fish.

QUESTION VIII.

If *A* can perform a piece of work in 12 days, and *B* can perform the same in 15 days, in what time will they finish it if they both work at it together?

Let x denote the number of days;

then $\frac{x}{12}$ is the part *A* can do in x days;

and $\frac{x}{15}$ is the part *B* can do in x days:

$$\therefore \frac{x}{12} + \frac{x}{15} = \text{the whole work } (= 1);$$

and clearing the equation, $15x + 12x = 180$;

that is, $27x = 180$:

$\therefore x = \frac{180}{27} = 6\frac{2}{3}$;

$\therefore$ they will finish it in $6\frac{2}{3}$ days.

9. A person wishes to divide a straight line into 3 parts, so that the first part may be 3 feet less than the second, and the second 5 feet more than the third. Required the length of each part, that of the whole line being 37 feet.

Ans. the three parts are 12, 15, and 10 feet.

10. What number is that whose fifth part exceeds its sixth by 7?

Ans. 210.

11. Two persons, at the distance of 150 miles, set out to meet each other: one goes 3 miles while the other goes 7. What part of the distance will each have travelled when they meet?

Ans. One 45 miles, and the other 105.

12. Divide the number 60 into two such parts, that their product may be equal to three times the square of the less.

Ans. The parts are 15 and 45.

13. Divide the number 45 into two parts, such that their product may be equal to the greater minus the square of the less.

Ans. The parts are $\frac{45}{46}$ and $\frac{2025}{46}$.

14. It is required to divide the number 36 into three such parts, that one half the first, one third the second, and one fourth the third, may be equal to each other. Ans. The parts are 8, 12, and 16.

15. A person bought three parcels of books, each containing the same number, for 12*l.* 5*s.*; for the first parcel he gave at the rate of 5*s.*, for the second 9*s.*, and for the third 10*s.* 6*d.* a volume. How many were there in each parcel? Ans. 10.

16. Find a number such that $\frac{1}{3}$ thereof increased by $\frac{1}{4}$ of the same shall be equal to $\frac{1}{6}$ of it increased by 35. Ans. 84.

17. A post is $\frac{1}{3}$ in mud, $\frac{1}{4}$ in water, and 10 feet above the water. Required the length of the post. Ans. 24 feet.

18. There is a cistern which can be supplied with water from three different cocks; from the first it can be filled in 8 hours, from the second in 10 hours, and from the third in 14 hours. In what time will it be filled if the three cocks be all set running together?

Ans. 3 hours, 22 min. $24\frac{48}{83}$ sec.

19. A gentleman spends $\frac{2}{3}$ of his yearly income in board and lodging, $\frac{2}{3}$ of the remainder in clothes, and lays by $20l.$ a year. What is his income? Ans. $180l.$

20. Two travellers set out at the same time, the one from London in order to travel to York, and the other from York to travel to London; the one goes 14 miles a day, and the other 16. In what time will they meet, the distance between London and York being 197 miles?

Ans. In 6 days, $13\frac{3}{5}$ hours.

21. A person wishes to give $3d.$ a piece to some beggars, but finds he has not money enough by $8d.$; but if he gives them $2d.$ a piece, he will have $3d.$ remaining. Required the number of beggars. Ans. 11.

22. A gamester at play staked $\frac{1}{5}$ of his money, which he lost, but afterwards won $4s.$; he then lost $\frac{1}{4}$ of what he had, and afterwards won $3s.$; after this he lost $\frac{1}{6}$ of what he then had, and finding that he had but $1l.$ remaining, he left off playing. It is required to find how much he had at first. Ans. $1l.\ 10s.$

23. A person mixed 20 gallons of spirits at $9s.$ a gallon with 36 gallons at $11s.$ a gallon, and he now wishes to add such a quantity at $14s.$ a gallon as will make the whole worth $12s.$ a gallon. How much of this last must he add? Ans. 48 gallons.

24. If A can perform a piece of work in a days, and B can do the same in b days, in how many days will they have finished the work if they both work at it together?

Ans. In $\dfrac{ab}{a+b}$ days.

25. If A can perform a piece of work in a days, B in b days, C in c days, and D in d days, in how many days will they have finished the work, if they all work at it together?

Ans. In $\dfrac{abcd}{abc+abd+bdc+adc}$ days.

26. It is required to find a number such that, if it be increased by 7, the square root of the sum shall be equal to the square root of the number itself, and 1 more. Ans. 9.

27. It is required to find two numbers, whose difference is 6, such that, if $\frac{1}{3}$ the less be added to $\frac{1}{5}$ the greater, the sum shall be equal to $\frac{1}{3}$ the greater diminished by $\frac{1}{5}$ the less. Ans. 2 and 8.

28. A labourer engages to work at the rate of 3*s*. 6*d*. a day, but on every day that he is idle he spends 9*d*., and at the end of 24 days finds that, upon deducting his expenses, he has to receive 3*l*. 2*s*. 9*d*. How many days was he idle? Ans. 5 days.

29. A person being asked the hour, answered that it was between 5 and 6, and that the hour and minute hands were exactly together. What was the time? Ans. 27′ $16\frac{4}{11}$″ past 5.

30. A gentleman leaves 315*l*. to be divided among his four sons in the following manner, viz. the second is to receive as much as the first and half as much more; the third is to receive as much as the first and second together and $\frac{1}{3}$ as much more; and the eldest is to receive as much as the other three and $\frac{1}{4}$ as much more. Required the share of each.

Ans. { The share of the 1st is 24*l*. of the 2d, 36*l*., of the 3d, 80*l*., and of the 4th, 175*l*.

PROBLEM IV.

To resolve Simple Equations containing two Unknown Quantities.

(54.) When there are given two independent simple equations, and two unknown quantities, the value of each unknown quantity may be obtained by either of the three following methods:

First method.

(55.) Find the value of one of the unknown quantities in terms of the other and the known quantities, from the first equation, by the method already given. Find the value of the same unknown quantity from the second equation.

Put these two values equal to each other, and we shall then have a simple equation containing only one unknown quantity, which may be solved as before.

Thus, suppose $ax + by = c$; and $a'x + b'y = c'$.

Then, from the first equation, $x = \dfrac{c - by}{a}$:

and from the second . . $x = \frac{c' - b'y}{a'}$;

whence, equating these two values of x, $\frac{c - by}{a} = \frac{c' - b'y}{a'}$;

and clearing the equation, $a'c - a'by = ac' - ab'y$;

or by transposition, $ab'y - a'by = ac' - a'c$;

$$\text{that is, } (ab' - a'b)\,y = ac' - a'c:$$

$$\therefore y = \frac{ac' - a'c}{ab' - a'b};$$

and this value being substituted in either of the above values of x gives

$$x = \frac{b'c - bc'}{ab' - a'b}.$$

EXAMPLES.

1. Given $\left\{\begin{array}{l} 2x + 3y = 23 \\ 5x - 2y = 10 \end{array}\right\}$, to find the values of x and y.

From the first equation $x = \frac{23 - 3y}{6}$,

and from the second ...$x = \frac{10 + 2y}{5}$:

$$\therefore \frac{23 - 3y}{2} = \frac{10 + 2y}{5};$$

$$\text{or } 115 - 15y = 20 + 4y;$$

and by transposition, $-15y - 4y = 20 - 115$;

$$\text{or } -19y = -95:$$

$$\therefore y = \frac{-95}{-19} = 5;$$

$$\text{consequently, } x \left(= \frac{10 + 2y}{5}\right) = 4.$$

2. Given $\left\{\begin{array}{l} 5x + 2y = 45 \\ 4x + y = 33 \end{array}\right\}$, to find the values of x and y.

From the first equation, $y = \dfrac{45 - 5x}{2}$,

and from the second ... $y = 33 - 4x$:

$$\therefore \frac{45 - 5x}{2} = 33 - 4x,$$

$$\text{or } 45 - 5x = 66 - 8x;$$

and by transposition, $8x - 5x = 66 - 45$;

that is, $3x = 21$:

$$\therefore x = 7, \text{ and } y\ (= 33 - 4x) = 5;$$

3. Given $\left\{ \begin{array}{l} 6x - 5y = 39 \\ 7x - 3y = 54 \end{array} \right\}$, to find the values of x and y.

Ans. $x = 9$ and $y = 3$.

3. Given $\left\{ \begin{array}{l} \frac{1}{4}x + \frac{1}{2}y = 7 \\ \frac{1}{3}x - \frac{1}{4}y = 2 \end{array} \right\}$, to find the values of x and y.

Ans. $x = 12$ and $y = 8$.

5. Given $\left\{ \begin{array}{r} 3x - \frac{1}{4}y = 7 \\ -\frac{1}{2}x + 2y = 14\frac{1}{2} \end{array} \right\}$, to find the values of x and y.

Ans. $x = 3$ and $y = 8$.

Second method.

(56.) Find the value of either of the unknown quantities from one of the equations, as in the preceding method. Substitute this value for its equal in the other equations, and we shall have an equation containing only one unknown quantity.

Thus, taking the same general example as before, viz. $ax + by = c$, and $a'x + b'y = c'$, if we substitute for x in the second equation its value, $\dfrac{c - by}{a}$, as determined from the first, there will arise the equation

$$\frac{a'c - a'by}{a} + b'y = c'; \text{ or } a'c - a'by + ab'y = ac';$$

and by transposition, $ab'y - a'by = ac' - a'c$:

$$\left.\begin{array}{r} \therefore y = \dfrac{ac' - a'c}{ab' - a'b} \\ \text{and by substitution, } x = \dfrac{b'c - bc'}{ab' - a'b} \end{array}\right\} \text{as before.}$$

EXAMPLES.

1. Given $\left\{\begin{array}{l} 8x + 6y = 74 \\ 3x + 5y = 36 \end{array}\right\}$, to find the values of x and y.

From the first equation $8x = 74 - 6y$, or $x = \dfrac{74 - 6y}{8}$;

which value substituted in the second equation,

$$\text{gives } \frac{222 - 18y}{8} + 5y = 36:$$

$$\therefore 222 - 18y + 40y = 288,$$

$$\text{or } 40y - 18y = 288 - 222;$$

$$\text{that is, } 22y = 66:$$

$$\therefore y = \tfrac{66}{22} = 3, \text{ and } x \left(= \frac{74 - 6y}{8}\right) = 7.$$

2. Given $\left\{\begin{array}{l} 7x + 2y = 30 \\ 5x + 3y = 34 \end{array}\right\}$, to find the values of x and y.

Ans. $x = 2$ and $y = 8$.

3. Given $\left\{\begin{array}{l} \frac{1}{2}x + \frac{1}{3}y = 8 \\ \frac{1}{3}x - \frac{1}{2}y = 1 \end{array}\right\}$, to find the values of x and y.

Ans. $x = 12$ and $y = 6$.

4. Given $\left\{\begin{array}{l} \dfrac{2x + 3y}{4} = 5 \\ 2x = \dfrac{54 - 8y}{3} \end{array}\right\}$, to find the values of x and y.

Ans. $x = 1$ and $y = 6$.

Third method.

(57.) Multiply each of the given equations by such quantities that the coefficient of one of the unknown quantities may be the same in both.

Destroy the identical terms by adding or subtracting these equations, and the result will be an equation containing only one unknown quantity.

NOTE. If multipliers do not readily present themselves, which will make the coefficient of any one of the unknowns the same in both equations, then each of the equations must be multiplied by the coefficient of that unknown in the other equation, which we wish to exterminate.

Thus, taking our former general example, $ax + by = c$, and $a'x + b'y = c'$; if we multiply the second equation by a and the first by a', in order that the coefficient of x may be the same in both equations, we shall have

$$aa'x + ab'y = ac';$$

$$aa'x + a'by = a'c;$$

and subtracting, $ab'y - a'by = ac' - a'c$

$$\left.\begin{array}{r} \therefore y = \dfrac{ac' - a'c}{ab' - a'b} \\ \text{and by a similar process, } x = \dfrac{b'c - bc'}{ab' - a'b} \end{array}\right\}\text{as before.}$$

EXAMPLES.

1. Given $\left\{\begin{array}{l} 4x - 3y = 1 \\ 3x + 4y = 57 \end{array}\right\}$, to find the values of x and y.

Multiplying the first equation by 3 and the second by 4, in order to equalize the coefficients of x, we have

$$12x - 9y = 3$$

$$12x + 16y = 228$$

and subtracting, $25y = 225$:

$$\therefore y = \tfrac{225}{25} = 9;$$

$$\text{whence } x = \frac{3 + 9y}{12} = \frac{3 + 81}{12} = 7.$$

2. Given $\left\{\begin{array}{l} 6x + 5y = 128 \\ 3x + 4y = \ \ 88 \end{array}\right\}$, to find the values of x and y.

Ans. $x = 8$ and $y = 16$.

3. Given $\left\{\begin{array}{r} 7x + 3y = 42 \\ -2x + 8y = 50 \end{array}\right\}$, to find the values of x and y.

Ans. $x = 3$ and $y = 7$.

ADDITIONAL EXAMPLES.

1. Given $\left\{\begin{array}{l} 5x + 7y = 201 \\ 8x - 3y = 137 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 22 \\ y = 13 \end{array}\right.$

2. Given $\left\{\begin{array}{l} -3x + 8y = 29 \\ -4x + 6y = 20 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 1 \\ y = 4 \end{array}\right.$

3. Given $\left\{\begin{array}{r} 3x - \frac{1}{2}y = \ 3\frac{1}{2} \\ -\ \ x + 7y = 33 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 2 \\ y = 5 \end{array}\right.$

4. Given $\left\{\begin{array}{l} \frac{1}{2}x + \frac{1}{3}y = \ \ \ 8 \\ \frac{1}{3}x - \frac{1}{5}y = -1 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = \ 6 \\ y = 15 \end{array}\right.$

5. Given $\left\{\begin{array}{l} \dfrac{2x}{3} + 5y = \ \ \ 23 \\ 5x + \dfrac{7y}{4} = -6\frac{1}{4} \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = -3 \\ y = \ \ \ 5 \end{array}\right.$

6. Given $\left\{\begin{array}{l} \dfrac{x}{2} - 12 = \dfrac{y}{4} + 8 \\ \dfrac{x+y}{5} + \dfrac{x}{3} - 8 = \dfrac{2y-x}{4} + 27 \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 60 \\ y = 40 \end{array}\right.$

7. Given $\left\{\begin{array}{l} x+y=a \\ x^2-y^2=b \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=\dfrac{a^2+b}{2a} \\ y=\dfrac{a^2-b}{2a} \end{array}\right.$

8. Given $\left\{\begin{array}{l} b(x+y)=a(x-y) \\ x^2-y^2=c \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=\dfrac{a+b}{2}\sqrt{\dfrac{c}{ab}} \\ y=\dfrac{a-b}{2}\sqrt{\dfrac{c}{ab}} \end{array}\right.$

(58.) QUESTIONS PRODUCING SIMPLE EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES.

QUESTION I.

A vintner sold, at one time, 20 dozen of port wine and 30 dozen of sherry, and for the whole received 120*l.*; and at another time, he sold 30 dozen of port and 25 of sherry, at the same prices as before, and for the whole received 140*l.* What was the price of a dozen of each sort of wine?

Let x be the price of the port per dozen,
and y that of the sherry;

$$\left.\begin{array}{l} \text{then } 20x+30y=120 \\ \text{and } \ 30x+25y=140 \end{array}\right\} \text{ or } \left\{\begin{array}{l} 2x+3y=12 \\ 6x+5y=28 \end{array}\right\}$$

and multiplying the first equation by 3,

$$6x+9y=36$$

and subtracting $6x+5y=28$

$$4y=8$$

$\therefore y=2$; $\therefore$ 2*l.* is the price of the sherry;

and $x\left(=\dfrac{12-3y}{2}\right)=3$; $\therefore$ 3*l.* is the price of the port per dozen.

QUESTION II.

A farmer has 86 bushels of wheat at 4*s.* 6*d.* per bushel, with which he wishes to mix rye at 3*s.* 6*d.* per bushel, and barley at 3*s.* per bushel, so as to make 136 bushels that shall be worth 4*s.* a bushel. What quantity of rye and of barley must he take?

Let x represent the number of bushels of rye,

and y the number of barley;

then $3\frac{1}{2}x$ shillings is the value of the rye,

$3y$ shillings . . . . barley,

and 387 shillings . . . . wheat.

Now the value of the whole 136 bushels, at 4*s.*, is 544*s.*;

$$\therefore 3\tfrac{1}{2}x + 3y + 387 = 544;$$

$$\text{or } 3\tfrac{1}{2}x + 3y = 157;$$

also $x+y+86 = 136$; $\therefore 3x + 3y = 150$, by transposing and multiplying by 3;

$$\text{and by subtraction,} \quad \tfrac{1}{2}x = 7;$$

$$\therefore x = 14;$$

$$\text{and } y\ (= 136 - 86 - x) = 36:$$

hence he must take 14 bushels of rye,

and 36 . . . barley.

QUESTION III.

A person has 27*l.* 6*s.* in guineas and crown-pieces; out of which he pays a debt of 14*l.* 17*s.*, and finds he has exactly as many guineas left as he has paid away crowns, and as many crowns as he has paid away guineas. How many of each had he at first?

Suppose x the number of guineas paid away,

and y . . . . . crowns . . . . .;

then, by reducing to shillings, we have

$$\left.\begin{aligned} & 21x + 5y = 297 = \text{the amount paid away} \\ \text{and} \quad & 5x + 21y = 249 = \text{the amount remaining} \end{aligned}\right\} \text{by the question;}$$

∴ multiplying the first equation by 5 and the second by 21,

$$\text{we have} \begin{cases} 105x + 25y = 1485 \\ 105x + 441y = 5229 \end{cases}$$

$$\text{and by subtraction} \quad . \quad . \quad . \quad . \quad 416y = 3744$$

$$\therefore y = \tfrac{3744}{416} = 9 = \text{number of crowns paid away;}$$

$$\text{whence } x \left(= \frac{249 - 21y}{5}\right) = 12 = \text{number of guineas paid away;}$$

∴ he had at first 21 guineas and 21 crowns.

QUESTION IV.

There is a number, consisting of two digits, which is equal to four times the sum of those digits; and if 9 be subtracted from twice the number, the digits will be inverted. What is the number?

$$\text{Put } x = \text{the first digit,}$$
$$y = \text{the second;}$$

$$\left.\begin{aligned} \text{then the number is } 10x + y &= 4x + 4y \\ \text{also } 20x + 2y - 9 &= 10y + x \end{aligned}\right\} \text{by the question.}$$

From the first equation $6x = 3y$, or $y = 2x$,
and from the second $19x - 8y = 9$; or, substituting the above value of y in this equation, we have $19x - 16x = 9$, or $3x = 9$;

$$\therefore x = 3 \text{ and } y\ (= 2x) = 6, \therefore \text{the number is } 36.$$

5. A bill of 14*l.* 8*s.* was paid with half-guineas and crowns, and twice the number of crowns was equal to three times the number of half-guineas. How many were there of each?

Ans. 16 half-guineas and 24 crowns.

6. There is a number consisting of two digits, which is equal to four times the sum of those digits; and if 18 be added to it, the digits will be inverted. What is the number? Ans. 24.

7. A man being asked the age of himself and son replied, "If I were $\frac{1}{4}$ as old as I am + 3 times the age of my son, I should be 45; and if he were $\frac{1}{4}$ his present age + 3 times mine, he would be 111." Required their ages.

Ans. The father's age was 36, and the son's 12.

8. What fraction is that, whose numerator being doubled and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled and the numerator increased by 2, the value becomes $\frac{3}{5}$?

Ans. $\frac{4}{5}$.

9. A man and his wife could drink a barrel of beer in 15 days; but, after drinking together 6 days, the woman alone drank the remainder in 30 days. In what time could either alone drink the whole barrel?

Ans. The man could drink it in $21\frac{3}{7}$ days; the woman in 50 days.

10. A farmer sold at one time 30 bushels of wheat and 40 bushels of barley, and for the whole received 13*l.* 10*s.*; and at another time he sold, at the same prices as before, 50 bushels of wheat and 30 bushels of barley, and for the whole received 17*l.* How much was each sort of grain sold at per bushel?

Ans. The wheat was sold at 5*s.* and the barley at 3*s.* a bushel.

PROBLEM V.

To resolve Simple Equations containing three Unknown Quantities.

(59.) Either of the three methods given for the resolution of equations with two unknown quantities may be extended to this case; but, as the last of the three will generally be found preferable to the others, we shall therefore give it as our

First Method.

(60.) Multiply or divide each of the first two equations by such quantities as will make the coefficients of one of the unknowns the same in both.

Destroy the identical terms, by adding or subtracting these

equations, and the result will be an equation containing only two unknown quantities.

Perform a similar process on the first and third, or on the second and third, of the original equations, and there will result another equation containing only two unknown quantities; therefore we shall have two equations and two unknown quantities: hence this problem is reduced to the former.

After what has been done in Art. 57, there does not seem any necessity for showing the truth of this method in general terms; we shall therefore proceed to particular examples.

EXAMPLES.

1. Given $\left\{\begin{array}{l} 2x + 4y - 3z = 22 \\ 4x - 2y + 5z = 18 \\ 6x + 7y - \ z = 63 \end{array}\right\}$, to find the values of x, y, and z.

Multiplying the first equa. by 2, $4x + 8y - 6z = 44$
and subtracting the second . . $4x - 2y + 5z = 18$

there results . . . . $10y - 11z = 26$ . . . . . [A]
Again, mult. the first equa. by 3, $6x + 12y - 9z = 66$
and subtracting the third . . . $6x + 7y - z = 63$

there results . . . . $5y - 8z = 3$
and mult. this result by 2 . . $10y - 16z = 6$
which, subtracted from equation [A] $10y - 11z = 26$

gives . . . . $5z = 20$

$\therefore z = 4$, $y\left(= \frac{3 + 8z}{5}\right) = 7$, and $x\left(= \frac{22 - 4y + 3z}{2}\right) = 3$.

2. Given $\left\{\begin{array}{l} 3x + 2y - 4z = 8 \\ 5x - 3y + 3z = 33 \\ 7x + \ y + 5z = 65 \end{array}\right\}$, to find the values of x, y, and z.

In this example it appears, from the coefficients, that y may be most readily exterminated:

$\therefore$ multiplying the first equation by 3 and the second by 2, they become

$$\begin{aligned} 9x + 6y - 12z &= 24 \\ 10x - 6y + 6z &= 66 \end{aligned}$$

and by addition . . . $19x - 6z = 90$ . . . . [A]

Again, multiplying the third equation by 2, it becomes

$$14x + 2y + 10z = 130$$

and subtracting the first, $3x + 2y - 4z = 8$

there results . . . . $11x + 14z = 122$

and multiplying this last equation by 3 and equation [A] by 7, we have

$$33x + 42z = 366$$

and . . $133x - 42z = 630$

and by addition, ... $166x = 996$

$$\therefore x = 6;$$

also $z \left(= \dfrac{90 - 19x}{-6}\right) = 4$, and $y \,(= 65 - 7x - 5z) = 3$.

3. Given $\left\{\begin{aligned} 7x + 5y + 2z &= 79 \\ 8x + 7y + 9z &= 122 \\ x + 4y + 5z &= 55 \end{aligned}\right\}$, to find the values of x, y, and z.

Ans. $\left\{\begin{aligned} x &= 4 \\ y &= 9 \\ z &= 3 \end{aligned}\right.$

4. Given $\left\{\begin{aligned} 3x - 9y + 8z &= 41 \\ -5x + 4y + 2z &= -20 \\ 11x - 7y - 6z &= 37 \end{aligned}\right\}$, to find the values of x, y and z.

Ans. $\left\{\begin{aligned} x &= 2 \\ y &= -3 \\ z &= 1 \end{aligned}\right.$

5. Given $\left\{\begin{aligned} x + \tfrac{1}{2}y + \tfrac{1}{3}z &= 32 \\ \tfrac{1}{3}x + \tfrac{1}{4}y + \tfrac{1}{5}z &= 15 \\ \tfrac{1}{4}x + \tfrac{1}{5}y + \tfrac{1}{6}z &= 12 \end{aligned}\right\}$, to find the values of x, y, and z.

Ans. $\left\{\begin{aligned} x &= 12 \\ y &= 20 \\ z &= 30 \end{aligned}\right.$

6. Given $\left\{\begin{array}{l} \frac{x+y}{3}+2z= \quad 21 \\ \frac{y+z}{2}-3x=-65 \\ \frac{3x+y-z}{2}= \quad 38 \end{array}\right\}$, to find the values of x, y, and z.

Ans. $\left\{\begin{array}{l} x=24 \\ y=\ 9 \\ z=\ 5 \end{array}\right.$

Second Method.

(61.) Multiply the first equation by some undetermined quantity m, and the second by another, n;

Add the two equations, so multiplied, together; and from the sum subtract the third equation, and the result will be an equation containing all the three unknown quantities.

Then determine m and n, so that the coefficients of two of the unknowns may become zero: these unknowns will thus disappear from the equation, leaving in it but a single unknown, the value of which may be immediately determined.

Thus, as a general example, let us take the three equations

$$\begin{aligned} ax+by+cz&=d, \\ a'x+b'y+c'z&=d', \\ a''x+b''y+c''z&=d''; \end{aligned}$$

then, multiplying the first by m, the second by n, adding the results, and subtracting the third equation, we have

$$(am+a'n-a'')x+(bm+b'n-b'')y+(cm+c'n-c'')z$$
$$=dm+d'n-d'' \quad .\ .\ .\ .\ [A];$$

now, in order to destroy x and y, put

$$\left.\begin{array}{r} am+a'n=a'' \\ \text{and } bm+b'n=b'' \end{array}\right\} \quad .\ .\ .\ .\ [B];$$

then, since the coefficients of x and y become 0, they vanish from the equation [A], which becomes simply

$$(cm+c'n-c'')z=dm+d'n-d'';$$

$$\therefore z=\frac{dm+d'n-d''}{cm+c'n-c''}$$

The values of m and n, being found from equations [B] by last problem, are

$$m=\frac{a''b'-b''a'}{ab'-ba'},$$

$$\text{and } n=\frac{ab''-ba''}{ab'-ba'};$$

and if these values be substituted in the above value of z, and the fractions, in both numerator and denominator, be brought to common denominators, we shall have

$$z=\frac{d\,(b'a''-a'b'')+d'\,(ab''-ba'')-d''\,(ab'-ba')}{c\,(b'a''-a'b'')+c'\,(ab''-ba'')-c''\,(ab'-ba')}.$$

In a similar manner may x and z be exterminated, and the value of y exhibited, by putting

$$am+a'n=a'',$$

$$cm+c'n=c'';$$

and y and z also may be exterminated, and the value of x exhibited, by putting

$$bm+b'n=b'',$$

$$cm+c'n=c'',$$

and proceeding as above. We shall therefore have

$$x=\frac{d\,(c'b''-b'c'')+d'\,(bc''-cb'')-d''\,(bc'-cb')}{a\,(c'b''-b'c'')+a'\,(bc''-cb'')-a''\,(bc'-cb')},$$

$$y=\frac{d\,(c'a''-a'c'')+d'\,(ac''-ca'')-d''\,(ac'-ca')}{b\,(c'a''-a'c'')+b'\,(ac''-ca'')-b''\,(ac'-ca')},$$

$$z=\frac{d\,(b'a''-a'b'')+d'\,(ab''-ba'')-d''\,(ab'-ba')}{c\,(b'a''-a'b'')+c'\,(ab''-ba'')-c''\,(ab'-ba')},$$

and, by substituting particular values in the above general expressions, any proposed example may be solved.

Whatever method we employ, in equations containing more than one unknown quantity, to remove or destroy these unknowns in succession, in order to bring out finally their individual values, as above, the process is called *Elimination;* and every unknown quantity thus removed is said to be *eliminated.*

SCHOLIUM.

(62.) The method above given has been introduced for the purpose of obtaining general values for the unknown quantities, that may apply to every particular example, by substituting in them the particular coefficients for the above general ones ; this method being preferable for that purpose to the preceding one. When, however, the whole process is to be performed, particular examples are much more readily solved by the first method, and therefore we shall not give any to this. It may here be further observed, that either of the two methods may be readily extended to equations containing four or a greater number of unknown quantities, they being solved, according to the first method, by equalizing the coefficients of the same unknown in any two equations, and then, by addition or subtraction, eliminating them one by one; or, according to the second method, by multiplying the first equation by m, the second by n, the third by p, &c. to the last but one, subtracting the last equation from their sum, and then determining m, n, p, &c., so that all the unknowns in the resulting expression may vanish, except one, the value of which will become known.

Both these methods, however, from their giving only one unknown at a time, and their requiring a repetition of the process to determine each of the others, become at length very tedious; a circumstance which has induced several eminent mathematicians to attempt the discovery of a direct method, whereby the values of all the unknowns, in any number of equations of this kind, may be determined at once. The most successful of these has been Bezout, who, first in the Memoirs of the Academy of Sciences, and then in his *Théorie Générale des Equations Algébriques*, p. 172, gave a method, which is generally considered as the simplest that has yet appeared. It is as follows:

General Rule to calculate, either all at once or separately, the values of the unknown quantities in equations of the first degree, whether they be literal or numeral.

Let u, x, y, z, &c. be the unknowns, whose number is n, this being also the number of the equations. Let a, b, c, d, &c. be the respective coefficients of the unknowns in the first equation; a', b', c', d', &c. the coefficients of those in the second; a'', b'', c'', d'', &c. of those in the third, &c.

Conceive the known term in each equation to be multiplied by some unknown quantity, represented by t; and form the product $uxyzt$, of all the unknowns written in any order at pleasure; but this order, once determined, is to be preserved throughout the operation.

Change, successively, each unknown in this product for its coefficient in the first equation, observing to change the sign of each even term: the result is called the *first line.*

Change, in this first line, each unknown for its coefficient in the second equation, observing, as before, to change the sign of each even term: the result is the *second line.*

Change, in this second line, each unknown for its coefficient in the third equation, still changing the sign of each even term, and the result is the *third line.*

Continue this process to the last equation, inclusively, and the last line that you obtain will give the values of the unknowns in the following manner:

Each unknown will have for its value a fraction, whose numerator will be the coefficient of the same unknown in the *last* or nth *line;* and the general denominator will be the coefficient of t, the unknown at first introduced.

Suppose, for example, we wish to find the values of x and y in the equations

$$ax + by + c = 0, \text{ and } a'x + b'y + c' = 0.$$

Introducing t, these equations become

$$ax + by + ct = 0, \text{ and } a'x + b'y + c't = 0;$$

and forming the product, xyt, and then changing x into a, y into b, t into c, and changing the signs of the even terms, we have, for the *first line*, $ayt - bxt + cxy$; then changing x into a', y into b', t into c', and changing the signs, as before, we have for the *second line*

$$ab't - ac'y - a'bt + bc'x + a'cy - b'cx, \text{ or}$$

$$(ab' - a'b)t - (ac' - a'c)y + (bc' - b'c)x;$$

$$\text{whence } x = \frac{bc' - b'c}{ab' - a'b}, \text{ and } y = \frac{-(ac' - a'c)}{ab' - a'b} = \frac{a'c - ac'}{ab' - a'b};$$

and in the same manner may this method be applied to any number of equations whatever, containing an equal number of unknown quantities.*

It is of importance to observe that, whatever method of solution be applied to equations involving two or more unknown quantities, no definite or consistent result can be obtained, unless the equations themselves are subject to certain restrictions, from which equations with only one unknown quantity are entirely free. These restrictions are twofold: the several conditions implied in the equations must be *compatible* with one another; and they must at the same time, be *independent* of one another. If the first requisite be wanting, it is clear that no solution can exist; since incompatible conditions are of course not to be fulfilled. If the second requisite be wanting, innumerable solutions exist; for since in such a case one of the equations is necessarily implied in one or more of the others, it follows that the equation so implied may be dispensed with, as a superfluous repetition of a statement already exhibited. In reality, therefore, the number of unknown quantities exceeds *by one* the number of independent or distinct equations. By the preceding rules, as many of these unknowns as there are equations may be found in terms of the coefficients and the remaining unknown quantity; and as to this latter unknown we may now give any value we please, it follows that there are innumerable sets of solutions. It would often be tedious and sometimes impossible to discover, from a mere contemplation of the equations themselves, whether they were all compatible and independent. It is the perfection of algebra, however, to make known to us, by indications of its own, whatever incon-

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* Bézout does not give any demonstration of this rule in the work above referred to, and seems to have obtained it by induction. But a demonstration from LAPLACE may be seen in GARNIER's *Analyse Algébrique*.

sistency or ambiguity may be concealed among the fundamental conditions from which its reasonings proceed; so that before the application of the ordinary rules, there requires no preparatory examination as to whether the individual case proposed may not be a case of ambiguity, or even one of absurdity. The same process which leads us to the true definite values, when such exist, will also lead us to unequivocal indications of the inconsistency or insufficiency of the proposed equations in all other cases.

As an example of *incompatible* conditions, let the equations

$$3x - 2y + 5z = 14$$
$$2x + y - 8z = 10$$
$$8x - 3y + 2z = 35$$

be proposed. Then, without seeking to discover the inconsistency, let us proceed by either of the foregoing methods to eliminate y; and we shall be led to the two equations

$$7x - 11z = 34, \quad 14x - 22z = 65,$$

and thence to the absurdity $3 = 0$.

We conclude at once, therefore, that the proposed equations are contradictory or incompatible; and accordingly we find, upon examination, that if we multiply each side of the first equation by 2, and add the products to the second equation, there will arise the first member of the third equation equal to 38; whereas that first member, by the third equation, is equal to 35; so that the equations involve contradictory conditions.

As an example of *insufficient* conditions, take the equations

$$3x - 2y + 5z = 14$$
$$2x + y - 8z = 10$$
$$6x - 4y + 10z = 28$$

The elimination of y conducts to the two equations

$$7x - 11z = 34, \quad 0 = 0,$$

the second showing that the first and third of the proposed equations do not in reality differ from each other.

From the equation just deduced, $x = \frac{34 + 11z}{7}$; substituting

this in either the first or third of the proposed equations, $y = \frac{2+34z}{7}$. Now, in whichever of the proposed equations we substitute these values of x and y, for the purpose of determining z, we shall invariably obtain for z the expression $z = \frac{0}{0}$, which is the ordinary symbol of *indeterminateness;* and the form under which the final results always appear when the solutions are innumerable.

This form may, however, present itself in an algebraical result, from other causes than the want of independence in the conditional equations. Thus in a general investigation, in which the coefficients, as well as the unknown quantities, are represented by letters, it may happen that a factor, involving some of these coefficients, may have been introduced, in the course of the solution, so as to occur in numerator and denominator of the final result, and cause both to vanish in certain hypotheses, or when the coefficients are replaced by their numerical values. For instance, the result $x = \frac{(a-b)\,p}{(a^2-b^2)\,q}$, when $a = b$, takes the form $x = \frac{0}{0}$; the original conditions, however, which have led to this result may have been so restricted as to render multiple values of x inadmissible, in which case the factor $a - b$, common to numerator and denominator in the general expression for x, and from which, in the proposed hypothesis, the form $\frac{0}{0}$ entirely arises, must have been introduced in the course of the reductions by the aid of which that expression has been obtained. Expunging then this extraneous factor from numerator and denominator, we have, for the single and definite value of x, when $a = b$,

$$x = \frac{p}{(a+b)\,q} = \frac{p}{2aq}$$

In all such general investigations, therefore, the occurrence of the form $\frac{0}{0}$ should send us to an examination of the original

equations, in order to ascertain whether the hypothesis which converts the general result into this form causes at the same time conditions to disappear, or two or more equations to become identical. If such be not the case, the solutions, so far from being innumerable, may, like as in the first example above, be impossible.

But in all *particular examples*, such as those just given, the indications of impossibility and of coincident conditions are always unequivocal and immediately intelligible.

(63.) QUESTIONS PRODUCING SIMPLE EQUATIONS INVOLVING THREE UNKNOWN QUANTITIES.

QUESTION I.

If A and B can perform a piece of work in 8 days, A and C together in 9 days, and B and C together in 10 days: in how many days can each alone perform the same work?

Let the number of days be x, y, and z, respectively;

$$\text{then } A \text{ can do } \frac{1}{x} \text{ of the whole in a day;}$$

$$B \dots \frac{1}{y} \dots\dots\dots ;$$

$$C \dots \frac{1}{z} \dots\dots\dots ;$$

and since A and B do the whole in 8 days;

$$\therefore \frac{8}{x} + \frac{8}{y} \text{ (= the whole work)} = 1;$$

also, since A and C do the same in 9 days,

$$\therefore \frac{9}{x} + \frac{9}{z} \text{ (= the whole work)} = 1;$$

and in the same manner,

$$\frac{10}{y} + \frac{10}{z} = 1;$$

$\therefore$ dividing the first of these equations by 8, the second by 9, and the third by 10, we have

$$\left.\begin{array}{l}\dfrac{1}{x}+\dfrac{1}{y}=\frac{1}{8}\\[2ex]\dfrac{1}{x}+\dfrac{1}{z}=\frac{1}{9}\\[2ex]\dfrac{1}{y}+\dfrac{1}{z}=\frac{1}{10}\end{array}\right\}\ldots\ldots[1]$$

and, subtracting the second equation from the first,

$$\frac{1}{y}-\frac{1}{z}=\tfrac{1}{8}-\tfrac{1}{9};$$

and adding the third to this, we get

$$[2]\ \ldots\ldots\ \frac{2}{y}=\tfrac{1}{8}-\tfrac{1}{9}+\tfrac{1}{10}=\tfrac{41}{360};\ \therefore y=\tfrac{720}{41}=17\tfrac{23}{41};$$

also, subtracting the third equation from the second, we have

$$\frac{1}{x}-\frac{1}{y}=\tfrac{1}{9}-\tfrac{1}{10};\ \therefore\frac{1}{x}=\tfrac{1}{9}-\tfrac{1}{10}+\tfrac{41}{720}=\tfrac{49}{720};$$

whence $x=\frac{720}{49}=14\frac{34}{49}$, and $\dfrac{1}{z}=\frac{1}{9}-\frac{49}{720}=\frac{31}{720}$; $\therefore z=\frac{720}{31}=23\frac{7}{31}$;

hence *A* can do the work in $14\frac{34}{49}$ days, *B* in $17\frac{23}{41}$ days, and *C* in $23\frac{7}{31}$ days.

These values might have been deduced, and perhaps somewhat more readily, by subtracting in succession each of the equations [1] from the sum of the other two; like as [2] was deduced above. If the time had been required in which the work would have been done by A, B, C conjointly, then representing this time by w, we should have had, besides [1], a fourth equation,

$$\frac{w}{x}+\frac{w}{y}+\frac{w}{z}=1\ \text{(the whole work)}$$

$$\therefore w=1\div\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

But, by adding equations [1] together,

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{8}+\frac{1}{9}+\frac{1}{10}$$

$$\therefore\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}\left(\frac{1}{8}+\frac{1}{9}+\frac{1}{10}\right)$$

$$\therefore w=\frac{2}{\frac{1}{8}+\frac{1}{9}+\frac{1}{10}}=\frac{720}{121}=5\tfrac{115}{121}\text{ days.}$$

It is worthy of notice that the numerator of w is the same as that common to the values of x, y, and z; and that the denominator is the sum of the denominators of those values; and such would always be the case whatever given numbers were proposed; as will appear by putting for the numbers the general values a, b, c.

2. It is required to find three numbers such that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall together make 46; $\frac{1}{3}$ of the first $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third, shall together make 35; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third, shall together make $28\frac{1}{3}$.

Ans. 12, 60, and 80.

3. The trinomial expression $ax^2 + bx + c$ is such that when 4 is put for x its value is 42; when 3 is put for x its value is 22; and when 2 is put for x its value is 8. Required the values of the coefficients a, b, c.

Ans. $a = 3$, $b = -1$, $c = -2$.

4. A sum of money was divided among four persons, in such a manner that the share of the first was $\frac{1}{2}$ the sum of the shares of the other three, the share of the second $\frac{1}{3}$ the shares of the other three, and the share of the third $\frac{1}{4}$ the shares of the other three; and it was found that the share of the first exceeded that of the last by 14*l.* What was the sum divided, and how much was each person's share?

Ans. The whole sum was 120*l.*; also the share of the first person was 40*l.*, of the second 30*l.*, of the third 24*l.*, and of the fourth 26*l.*

5. A person has 22*l.* 14*s.* in crowns, guineas, and moidores; and he finds that if he had as many guineas as crowns, and as many crowns as guineas, he should have 36*l.* 6*s.*; but if he had as many moidores as crowns, and as many crowns as moidores, he should have 45*l.* 16*s.* How many of each did he have?

Ans. 26 crowns, 9 guineas, and 5 moidores.

6. There are three numbers, which, taken two and two, give the sums a, b, c. What are the numbers?

Ans. $\frac{1}{2}(a + b - c)$, $\frac{1}{2}(a + c - b)$, $\frac{1}{2}(b + c - a)$.

7. There are four numbers, which, taken three and three, give the products a, b, c, d. What are the numbers?

$$\text{Ans.} \begin{cases} \dfrac{\sqrt[3]{abcd}}{a}, & \dfrac{\sqrt[3]{abcd}}{b}, \\[2ex] \dfrac{\sqrt[3]{abcd}}{c}, & \dfrac{\sqrt[3]{abcd}}{d}. \end{cases}$$

CHAPTER III.

ON RATIO, PROPORTION, AND PROGRESSION.

RATIO.

(64.) Ratio is usually defined to be the relation which one quantity bears to another of the same kind, with respect to magnitude; and as such relation may be expressed either by stating how much one exceeds the other, or how often one contains the other, ratio has accordingly been divided into two kinds—arithmetical ratio and geometrical ratio.

Arithmetical ratio is that which expresses the *difference* of the quantities compared.

(65.) Geometrical ratio expresses the *quotient* arising from the division of the quantities compared.

Thus, if a and b be compared, $b-a$ expresses their arithmetical ratio, and $\frac{b}{a}$ their geometrical ratio;* but, to prevent confusion, the

* In expressing the geometrical ratio of two quantities, it matters not whether the second term be divided by the first, as is done here, or the first term by the second; but whichever way is fixed upon, that must be preserved. It is however usual, when ratios of different magnitudes are compared, to express them by the division of the first term by the second; thus, the ratio of 4 to 2 is said to be greater than that of 4 to 3, because $\frac{4}{2}$ is greater than $\frac{4}{3}$; but, in the investigation of properties, the way used in the text is preferable. It is common, however, for writers on this subject to define ratio as the quotient of the antecedent by the consequent, to conform to this definition in establishing the fundamental properties of proportion, and then to depart from it as soon as they enter upon the doctrine of progression. In the geometrical series at page 98, r is affirmed to be the common ratio by all authors on the subject; whereas, according to the definition adverted to, $\frac{1}{r}$ and not r is the ratio.

term *ratio* is generally confined to the latter sense; and, instead of *arithmetical ratio*, the simple term *difference* is used; so that in what follows ratio always means geometrical ratio. Also, to avoid the too frequent repetition of the term, two dots are usually placed between the quantities to represent their ratio; thus, $a : b$ signifies the ratio of a to b; and a and b are called the *terms* of the ratio.

(66.) The first term in a ratio is called the *antecedent*, and the other the *consequent*.

(67.) In any number of ratios, if the antecedents and consequents be respectively multiplied together, the ratio of the products is said to be *compounded* of the preceding ratios: thus, in the following ratios, $a : b$, $c : d$, $e : f$, the product of the antecedents is ace, and that of the consequents bdf, and $ace ; bdf$ is the *compound* ratio.

(68.) If the antecedents and consequents be respectively the same in each of the simple ratios, as $a : b$, $a : b$, $a : b$, &c., then the compound ratio is $a^2 : b^2$, or $a^3 : b^3$, &c., according to the number of simple ratios; in which case $a^2 : b^2$ is sometimes called the *duplicate* ratio of $a : b$, $a^3 : b^3$ the *triplicate* ratio, &c.; also, $\sqrt{a} : \sqrt{b}$ is called the *sub-duplicate*, $\sqrt[3]{a} : \sqrt[3]{b}$ the *sub-triplicate*, &c.

(69.) If each antecedent in the simple ratios be the same as the consequent in the preceding, as $a : b$, $b : c$, $c : d$, &c., then the compound ratio abc &c. : bcd &c. is evidently the same as $a : d$, because $\frac{bcd}{abc} = \frac{d}{a}$, d being supposed here to be the last consequent. The ratio also evidently continues the same if each term be either multiplied or divided by any quantity.

ARITHMETICAL PROPORTION AND PROGRESSION.

(70.) If there be four quantities such that the difference of the first and second is the same as that of the third and fourth, these quantities are said to be in *arithmetical proportion*.

(71.) If there be any number of quantities such that the difference of the first and second, of the second and third, of the third and fourth, &c. are all equal, these quantities are said to be in *arithmetical progression*; and the progression is said to be increasing or decreasing, according as the successive terms increase or decrease.

(72.) THEOREM 1. If four quantities be in arithmetical *proportion*, the sum of the extremes is equal to the sum of the means.

For let $a, b, c, d,$ be in arithmetical proportion; then $b-a=d-c$; add $a+c$ to each side of this equation, and there results $b+c=a+d$.

THEOREM 2. If three quantities be in arithmetical progression, the sum of the extremes is equal to twice the mean.

For let $a, b, c,$ be the three quantities; then $b-a=c-b$; add $b+a$ to each side of this equation, and there results $2b=a+c$.

THEOREM 3. In any series of quantities in arithmetical progression, the sum of the two extremes is equal to the sum of any two terms equally distant from the extremes; or it is equal to twice the middle term, when the number of terms is odd.

For let a be the first term in the series, and d the common difference; then, if the series be increasing, it is $a, a+d, a+2d, a+3d, a+4d,$ &c., in which, if the first and fifth be considered as extremes, we have

$$a+(a+4d)=(a+d)+(a+3d)=2(a+2d);$$

and the same may be shown for any greater number of terms; as also when the series is decreasing, by changing the sign of d.

THEOREM 4. In any increasing arithmetical progression, the last term is equal to the first term *plus* the product of the common difference, and number of terms *less* one; but if the progression be decreasing, then the last term is equal to the first term *minus* the same product.

Let a be the first term, and d the common difference: then the increasing series is $a, a+d, a+2d, a+3d$, &c., and the decreasing series is $a, a-d, a-2d, a-3d$, &c., where it is obvious that any term in the first series consists of the first term a, *plus* as many times d as are equal to the number of terms preceding the proposed term; and any term in the second series consists of the first term a, *minus* as many times d as are equal to the number of terms preceding; therefore the nth term of the former series is $a+(n-1)d$, and of the latter $a-(n-1)d$.

THEOREM 5. The sum of any series of quantities in arithmetical progression is equal to the sum of the extremes multiplied by half the number of terms.

Let $a+(a+d)+(a+2d)+(a+3d)+$ &c., be the progression; then, if the number of terms be represented by n, the last term will be $a+(n-1)d$ (Theo. 4); and therefore, by reversing the terms, the same series may be written thus:

$$\{a+(n-1)d\}+\{a+(n-2)d\}+\{a+(n-3)d\}+\dots\dots \{a+(n-n)d\},$$

and adding this series to its equal, as expressed above,

$$\{2a+(n-1)d\}+\{2a+(n-1)d\}+\{2a+(n-1)d\}+\dots \{2a+(n-1)d\}=$$

twice the sum of the progression; and as there must be n terms in this last, as well as in the proposed series, and since each term is $2a+(n-1)d$, twice the sum must be $n\{2a+(n-1)d\}$; and the sum $=\frac{1}{2}n\{2a+(n-1)d\}$; that is, the expression for the sum S is

$$S=\tfrac{1}{2}n\{2a+(n-1)d\}$$

$$\text{or } S=\tfrac{1}{2}n\{a+\text{last term}\}$$

and this formula is obviously quite sufficient to enable us to determine any one of the quantities a, d, n, S, or *last term*, when the others are given. If, however, either the first or the last term, the common difference, and the sum, be given to find the number of terms, the solution will require a quadratic equation; and the same will happen if either of the extreme terms be given, in conjunction with the sum and difference, to find the other extreme term. These cases will be considered in Chapter V.

EXAMPLES.

1. Required the sum of 10 terms of the progression 1, 4, 7, 10, &c.

Here $a=1$, $d=3$, $n=10$, and l (last term) $=a+9d=28$,

$$\therefore \frac{n(a+l)}{2}=\frac{290}{2}=145, \text{ the sum required.}$$

2. The first term of an arithmetical progression is 14, and the sum of eight terms 28. What is the common difference?

Here the given quantities are, $a=14$, $n=8$, and $S=28$. Hence, making these substitutions in the general expression for S, we have

$$28=4\{28+7d\}=112+28d;$$

$$\therefore d=\frac{28-112}{28}=-3;$$

therefore the common difference is -3, and, consequently, the series is $14, 11, 8, 5, 2, -1, -4, -7$, &c.

3. An arithmetical series consisting of six terms has 8 for the first term and 23 for the last. Required the intermediate terms.

The expression for the last term l is $l=a+(n-1)d$, and in the question, a, l and n are given to find d; that is, we have the equation

$$23=8+5d; \therefore d=\frac{23-8}{5}=3;$$

hence, the first term being 8, and the common difference 3, the series must be 8, 11, 14, 17, 20, 23, where the four intermediate terms are exhibited. In this manner we may insert any proposed number of *arithmetical means* between two given numbers.

4. Required the sum of 100 terms of the series 1, 3, 5, 7, 9, &c.

Ans. 10000.

5. Required the sum of a decreasing arithmetical series, whose first term is 12, and the common difference of the terms $\frac{1}{2}$.

Ans. 150.

6. Required the sum of 25 terms of an arithmetical progression, whose first term is $\frac{1}{2}$, and the common increase of each term $\frac{1}{2}$.

Ans. $162\frac{1}{2}$.

7. Insert three arithmetical means between $\frac{1}{3}$ and $\frac{1}{2}$.

The means are $\frac{3}{8}$, $\frac{5}{12}$, $\frac{11}{24}$.

8. The first term of an arithmetical series is 1, the number of terms 23. What must the common difference be in order that the sum may be $149\frac{1}{2}$?

Ans. $\frac{1}{2}$.

GEOMETRICAL PROPORTION AND PROGRESSION.

PROPORTION.

(73.) If there be four quantities, such that the ratio of the first and second is the same as that of the third and fourth, these quantities are said to be in *geometrical proportion*. Thus, if $\frac{b}{a} = \frac{d}{c}$, then a, b, c, d, are in geometrical proportion; and this proportion is represented thus, $a : b :: c : d$, which is read, a is to b as c to d, or as a is to b so is c to d.

Hence, since if $\frac{b}{a} = \frac{d}{c}$, $\frac{b^n}{a^n} = \frac{d^n}{c^n}$, then $a^n : b^n :: c^n : d^n$; that is, if four quantities be proportional, then the same powers or roots of the four quantities are also in proportion.

(74.) THEOREM 1. If four quantities be proportional, the product of the extremes is equal to that of the means.

Let a, b, c, d, be the proportionals, then $\frac{b}{a} = \frac{d}{c}$; multiply each side by ac, and there results $bc = ad$.

THEOREM 2. If the product of two quantities be equal to the product of two others, then a proportion may be formed of the four quantities.

Let $qr = ps$; then, dividing each side by rp, there results $\frac{q}{p} = \frac{s}{r}$; $\therefore p : q :: r : s$.

THEOREM 3. If four quantities be proportional, they are also proportional when taken inversely; that is, when the consequents are made antecedents and the antecedents consequents.

For if $a : b :: c : d$, then (Theor. 1) $ad = bc$, and, consequently, (Theor. 2) $b : a :: d : c$.

THEOREM 4. If four quantities be proportional, they are proportional also when taken alternately; that is, the first is to the third as the second is to the fourth.

For if $a : b :: c : d$, then $\frac{b}{a} = \frac{d}{c}$; and multiplying by $\frac{c}{b}$, there results $\frac{c}{a} = \frac{d}{b}$, $\therefore a : c :: b : d$.

THEOREM 5. In any proportion, the first term is to the second *plus* or *minus* m times the first, as the third is to the fourth *plus* or *minus* m times the third.

Let $\frac{b}{a} = \frac{d}{c}$, then $\frac{b}{a} \pm m = \frac{d}{c} \pm m$, or $\frac{b \pm am}{a} = \frac{d \pm cm}{c}$; that is, $a : b \pm am :: c : d \pm cm$.

Corollary 1. Also, since, $a : c :: b \pm am : d \pm cm$ (Theor. 4); $\therefore$ $\frac{c}{a} = \frac{d \pm cm}{b \pm am}$; but $\frac{c}{a} = \frac{d}{b}$; $\therefore b : d :: b \pm am : d \pm cm$, and $b : b \pm am$ $:: d : d \pm cm$; that is, the second term is to the second *plus* or *minus* m times the first, as the fourth is to the fourth *plus* or *minus* m times the third; also, since $\frac{d + cm}{b + am} = \frac{d - cm}{b - am}$; each being equal to $\frac{c}{a}$,

$$\therefore b + am : d + cm :: b - am : d - cm.$$

Cor. 2. And if m be taken $= 1$, we shall then have

$$a : b \pm a :: c : d \pm c, \text{ and } b : b \pm a :: d : d \pm c;$$

likewise,

$$b + a : d + c :: b - a : d - c, \text{ or } b + a : b - a :: d + c : d - c;$$

that is, the sum of the first two terms is to their difference as the sum of the last two to their difference.

THEOREM 6. In any number of proportions, if all the corresponding antecedents and consequents be respectively multiplied together, the resulting products will be in proportion.

$$\text{Let} \begin{cases} \frac{b}{a} = \frac{d}{c} \text{ or } a : b :: c : d \\ \frac{f}{e} = \frac{h}{g} \ldots e : f :: g : h \\ \frac{k}{i} = \frac{m}{l} \ldots i : k :: l : m \end{cases}$$

&c. &c. &c.

Then, multiplying the corresponding sides of the above equations together, we have

$$\frac{bfk\ \&c.}{aei\ \&c.} = \frac{dhm\ \&c.}{cgl\ \&c.},\ \text{or}$$

$$aei\ \&c. : bfk\ \&c. :: cgl\ \&c. : dhm\ \&c.$$

THEOREM 7. In any number of equal ratios, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let the ratios be $\frac{b}{a} = \frac{d}{c} = \frac{f}{e} = \&c.$ Put $\frac{b}{a} = q$;

then $b = aq$, $d = cq$, $f = eq$, &c., and, by adding these equations together, $b + d + f + \&c. = aq + cq + eq + \&c.$

$$= q\,(a + c + e + \&c.):$$

$$\therefore \frac{b + d + f + \&c.}{a + c + e + \&c.} = q = \frac{b}{a} = \frac{d}{c} = \&c.,\ \text{or}$$

$$a : b :: a + c + e + \&c. : b + d + f + \&c.$$

Cor. 1. Hence, in any number of proportions, where the ratio of the first two and last two terms are respectively the same in each, the sums of the corresponding terms are in proportion.

Cor. 2. Hence, also, in two proportions of this kind, if the terms of one be subtracted from the corresponding terms of the other, the remainders will be in proportion; since the results are the same as if each of the terms subtracted were multiplied by -1, and added.

Cor. 3. Therefore, in any number of proportions, having the same equality of the ratios, if the corresponding terms of some be added, and those of others subtracted, the final results will still be in proportion.

THEOREM 8. In any number of proportions, if the sum or difference of the first and second terms, as also of the third and fourth, be respectively the same in each, then the sums of the corresponding terms are also in proportion.

Let the proportions be

$$a : b :: c : d$$
$$e : f :: g : h$$
$$i : k :: l : m$$
$$\&c.\ \&c.\ \&c.\ \&c.$$

then, by Theorem 5, Cor 2,

$$b \pm a : d \pm c :: a : c$$
$$f \pm e : h \pm g :: e : g$$
$$k \pm i : m \pm l :: i : l$$
&c. &c. &c.

Now, if the first and second terms, in each of these proportions, be respectively the same, then the ratio of the third and fourth terms will be the same in all; but (Theor. 4),

$$a : c :: b : d$$
$$e : g :: f : h$$
$$i : l :: k : m$$
&c. &c. &c.

∴ (Theorem 7, Cor. 1),

$$a + e + i + \&c. : c + g + l + \&c. :: b + f + k + \&c. : d + h + m + \&c.$$

or (Theorem 4),

$$a + e + i + \&c. : b + f + k + \&c. :: c + g + l + \&c. : d + h + m + \&c.$$

Schol. Corollaries similar to the last two of Theorem 7, may evidently be deduced from this theorem.

PROGRESSION.

(75.) A GEOMETRICAL PROGRESSION is a series of quantities, such, that the quotient of any one of them, and that which immediately precedes, is constantly the same; that is, each is in the same constant ratio to the next following, throughout the series.

Thus, the following is a geometrical progression, in which a is the first term, r the constant ratio, and n the number of terms:

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ ar^6\ \ldots\ ar^{n-1}.$$

From the bare inspection of this series, the following properties are obvious:

1. If any two terms be taken as extremes, their product is equal to that of any two terms equally distant from them; or, if the number of terms be odd, the product of the extremes is equal to

the square of the middle term; and hence a geometrical mean between two quantities is equal to the square root of their product.

2. The last term in any geometrical series is equal to the product of the first term, and that power of the ratio which is expressed by the number of terms, *minus* 1.

(76.) **Problem.** To find the sum s of any number of terms in a geometrical series.

Let $s = a + ar + ar^2 + ar^3 + ar^4 + \ldots\ldots ar^{n-2} + ar^{n-1}$;
then, multiplying each side by r, there results

$$sr = ar + ar^2 + ar^3 + ar^4 + ar^5 + \ldots\ldots ar^{n-1} + ar^n;$$

and subtracting the first equation from this, we have

$$sr - s = ar^n - a, \text{ whence } s = \frac{a(r^n - 1)}{r - 1};$$

or if the last term, ar^{n-1}, be represented by l, we have, by substitution,

$$s = \frac{rl - a}{r - 1}.$$

When, however, r is a proper fraction, and the series, which will then be a decreasing one, goes on to infinity, then the last term obviously becomes 0; and the expression for the sum is

$$s = \frac{a}{1 - r}.$$

Hence this rule:

(77.) Multiply the last term by the ratio, and divide the difference of this product and the first term by the difference between the ratio and unity; observing that in an infinite decreasing series the last term $= 0$.

EXAMPLES.

1. Required the sum of 9 terms of the series, 1, 2, 4, 8, 16, &c.

Here $a = 1$, $r = 2$, and $n = 9$, $\therefore ar^{n-1} = 2^8 = 256 =$ the last term; consequently, $\dfrac{256 \times 2 - 1}{2 - 1} = 511$, the sum required.

2. Required the sum of the series 1 $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c. continued to infinity.

Here $a = 1$, $r = \frac{1}{2}$, and $\therefore \frac{1}{1 - \frac{1}{2}} = 2$, the sum required.

3. Given the first term 3, the last term 768, and the number of terms 9, to find the common ratio.

Here $a = 3$, $l = 768$, and $n = 9$, and the general expression for the last term being ar^{n-1}, we have in the present case,

$$768 = 3r^8 \therefore r = 256^{\frac{1}{8}} = 2;$$

hence, the intermediate terms of the series are 6, 12, 24, 48, 96, 192, and 384: and in this way may any number of *geometric means* be interposed between any *two* given *extremes*.

4. Required the sum of 10 terms of the series 9, 27, 81, 243, &c.

Ans. 265716.

5. Required the sum of the series $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}$, &c. continued to infinity.

Ans. $\frac{2}{3}$.

6. Required the sum of $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$, &c. continued to 10 terms.

Ans. $1\frac{9841}{19683}$.

7. Required the sum of 6 terms of the series $1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} +$ &c.

Ans. $\frac{3367}{7168}$.

8. It is required to insert three geometric means between $\frac{1}{2}$ and $\frac{2}{9}$.

Ans. The means are $\frac{1}{2}\sqrt{\frac{2}{3}}$, $\frac{1}{3}$, and $\frac{1}{3}\sqrt{\frac{2}{3}}$.

9. Required the sum of the series $1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} +$ &c. to infinity.

Ans. $\frac{x}{x-1}$.

10. Insert three geometric means between the extremes 4 and 324.

Ans. 12, 36, 108.

11. Suppose a body to move eternally in this manner, viz. 20 miles the first minute, 19 miles the second, $18\frac{1}{20}$ the third, and so on in geometrical progression. Required the utmost distance it can reach.

Ans. 400 miles.

HARMONICAL PROPORTION.

(78.) Three quantities are said to be in harmonical proportion when the first has the same ratio to the third as the difference between the first and second has to the difference between the second and third.

(79.) And four quantities are in harmonical proportion when the first has the same ratio to the fourth as the difference between the first and second has to the difference between the third and fourth.

Thus the quantities a, b, c, are in harmonical proportion when $a : c :: a - b : b - c$; and a, b, c, d, are in harmonical proportion when $a : d :: a - b : c - d$.

(80.) From these definitions it follows that, in three harmonical proportionals, a, b, c, any two being given, the third may be found;

For, since $a : c :: a - b : b - c$, $\therefore ab - ac = ac - bc$,

$$\text{or } ab + bc = 2ac;$$

$$\therefore b = \frac{2ac}{a + c};$$

that is, a harmonical mean between two quantities is equal to twice their product divided by their sum.

Also, $c = \dfrac{ab}{2a - b}$ = a third harmonical proportion to a and b.

(81.) In a similar manner, if any three out of four harmonical proportionals, a, b, c, d, be given, the other may be found; for since

$$a : d :: a - b : c - d, \therefore ac - ad = ad - bd;$$

and from this equation we get

$$b = \frac{2ad - ac}{d}; \quad c = \frac{2ad - bd}{a}; \quad d = \frac{ac}{2a - b}.$$

When there is a series of quantities such that every three consecutive terms are in harmonical proportion, the series is called a harmonical progression. The reciprocals of the terms of every arithmetical series produce a harmonical series, and conversely, the reciprocals of a harmonical series form a series in arithmetical

progression. For let a, b, c, d, &c. be an arithmetical series, and let a', b', c', d', &c., be put for the reciprocals of its terms; then

$$a+c=2b,\ b+d=2c,\ \&c. \ldots\ [1]$$

and, dividing each equation by the product of the three letters which enter it, we have

$$\frac{1}{bc}+\frac{1}{ab}=\frac{2}{ac},\ \frac{1}{cd}+\frac{1}{bc}=\frac{2}{bd},\ \&c. \ldots\ [2]$$

that is

$$b'c'+a'b'=2a'c',\ c'd'+b'c'=2b'd'\ \&c. \ldots\ [3]$$

so that (80) a', b', c', &c. are in harmonical progression. And it is plain that we might pass from [3], implying that a', b', c', d', &c. are in harmonical progression, through [2] to [1]; and thence infer that the reciprocals of a', b', c', d', &c. are in arithmetical progression.

The term *harmonical* seems to have been applied to series of this kind from the circumstance that musical strings of equal thickness and tension, in order to produce perfect harmony when sounded together, must have their lengths as the reciprocals of the arithmetical series of natural numbers, that is, as $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c.

As in the former progressions, arithmetical and geometrical, so in the harmonical, we can always discover the intermediate terms when the first and last and the number of terms are given, that is, we can always interpose any assigned number of harmonical means between two given extremes. And the method of doing this is at once suggested by the relation just established between harmonical and arithmetical series; for we shall only have to take the reciprocals of the given extreme terms, to regard these reciprocals as the extremes of an arithmetical series, to insert between them the proposed number of terms, and then to take the reciprocals of those terms.

Thus to insert two harmonic means between 3 and 12 we have, by arithmetical progression (p. 94)

$$\tfrac{1}{12}=\tfrac{1}{3}+3d\ \therefore d=-\tfrac{1}{12}$$

$\therefore$ the arithmetical means are $\frac{1}{4}$, $\frac{1}{6}$, $\therefore$ the harmonical means are 4, 6; so that the harmonical progression is 3, 4, 6, 12. Upon the same principle the nth term may be found when the first and second are given.

(82.) QUESTIONS IN WHICH PROPORTION IS CONCERNED.

QUESTION I.

Find a number, such that, if 3, 8, and 17 be severally added thereto, the first sum shall be to the second as the second to the third.

Let x be the number;

$$\text{then } x+3 : x+8 :: x+8 : x+17;$$

and by Cor. 2, Theor. 5, Art. 74, we have

$$x+3 : 5 :: x+8 : 9,$$

$\therefore$ (Theor. 1, Art. 74), $9x+27=5x+40$,

$$\text{or } 4x=13;$$

$$\therefore x=\tfrac{13}{4}=3\tfrac{1}{4}, \text{ the number required.}$$

QUESTION II.

A person has British wine at 5s. per gallon, with which he wishes to mix spirits at 11s. per gallon, in such proportion that, by selling the mixture at 9s. a gallon, he may gain 35 per cent. What is the necessary proportion?

Let the proportion of the wine to the spirits be as $x:y$;

then $5x+11y=$ prime cost of $x+y$ gallons,

and $9x+9y=$ selling price . . . . .

$\therefore 4x-2y=$ profit . . . .

and by the question,

$$5x+11y : 4x-2y :: 100 : 35 :: 20 : 7 \text{ (Art. 73)};$$

$\therefore$ (Theor. 1, Art. 74,) $80x-40y=35x+77y$,

$$\text{or } 45x=117y;$$

$$\therefore 5x=13y;$$

whence (Theor. 2) $x:y::13:5$;

$\therefore$ the mixture must be at the rate of 13 gallons of wine to 5 gallons of spirits.

QUESTION III.

A merchant, having mixed a certain number of gallons of brandy and water, found that if he had mixed 6 gallons more of each there would have been 7 gallons of brandy to every 6 gallons of water; but if he had mixed 6 gallons less of each, there would have been 6 gallons of brandy to every 5 gallons of water. How much of each did he mix?

Let x be the number of gallons of brandy,

y the number of gallons of water;

then, by the question, $\begin{cases} x+6 : y+6 :: 7 : 6 \\ x-6 : y-6 :: 6 : 5 \end{cases}$

$\therefore$ (Theorem 8), $x : y :: 13 : 11 \ldots\ldots$ [A];

also by the first proportion (Th. 5), $x-y : 1 :: x+6 : 7$

and by the second . . . . $x-y : 1 :: x-6 : 6$

$\therefore$ (Theor. 5, Cor. 2), $x-6 : 6 :: 2x : 13$

and (Theor. 1), $13x - 78 = 12x$;

$\therefore x = 78$;

also by substitution [A], $78 : y :: 13 : 11$,

whence $13y = 858$;

$\therefore y = 66$;

consequently the mixture consisted of 78 gallons of brandy and 66 of water.

A much simpler solution to this question may be obtained as follows: Instead of representing the number of gallons of brandy by x, and the number of gallons of water by y, represent these quantities by $7x - 6$ and $6x - 6$, respectively; by which artifice the first condition in the question is at once fulfilled; so that we have only to express the second, viz.

$$7x - 12 : 6x - 12 :: 6 : 5,$$

or (Theor. 5, Cor. 2) $x : 6x - 12 :: 1 : 5$,

$$\text{whence } 5x = 6x - 12 \;\therefore\; x = 12;$$

$$\therefore\; 7x - 6 = 78 \text{ gallons of brandy.}$$

$$\text{and } 6x - 6 = 66 \text{ gallons of water.}$$

This neat solution is given in the American edition of this work, published by Mr. Ward, of Columbia College.

5. A corn-factor mixes wheat which cost 10*s*. a bushel with barley which cost 4*s*. a bushel, in such proportion as to gain $43\frac{3}{4}$ per cent. by selling the mixture at 11*s*. a bushel. What is the proportion?

Ans. There are 14 bushels of wheat to 9 of barley.

5. It is required to find a number, such that the sum of its digits is to the number itself as 4 to 13; and if the digits be inverted, their difference will be to the number expressed as 2 to 31.

Ans. Either 13, 26, or 39.

6. At a certain instant, between five and six o'clock, the hour and minute hands of a clock are exactly together. Required the time.

Ans. 27 minutes $16\frac{4}{11}$ seconds past 5.

7. Required two numbers, such that their sum, difference, and product may be as the numbers 3, 2, and 5, respectively,

Ans. 10 and 2.

8. There are two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, and such that if they be increased respectively by 6 and 5, they will be to each other as $\frac{2}{5}$ to $\frac{1}{2}$. What are the numbers?

Ans. 30 and 40.

9. A person has some choice brandy at 40*s*. 6*d*. per gallon, which he wishes to mix with other brandy at 36*s*. a gallon, in such proportion, that the compound may be worth 39*s*. 6*d*. a gallon. What must the proportion be?

Ans. 7 gallons of the best to 2 gallons of the other.

10. Find two numbers, such that their sum, difference, and product may be as the numbers s, d, and p, respectively.

Ans. $\frac{2p}{s+d}$ and $\frac{2p}{s-d}$.

11. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are as much as 3 of the hare's. How many leaps must the greyhound take to catch the hare?

Ans. 300.

12. If three agents, A, B, C, can produce the effects a, b, c, in the times e, f, g, respectively; in what time would they jointly produce the effect d?

Ans. $d \div (\frac{a}{e} + \frac{b}{f} + \frac{c}{g})$.

13. The sum of the first and third of four numbers in geometrical progression is 148, and the sum of the second and fourth is 888. What are the numbers?

Ans. 4, 24, 144, and 864.

14. A and B speculate in trade with different sums. A gains 150*l.*, B loses 50*l.*; and now A's stock is to B's as 3 to 2; but, had A lost 50*l.* and B gained 100*l.*, then A's stock would have been to B's as 5 to 9. What was the stock of each?

Ans. A's was 300*l.* and B's 350*l.*

CHAPTER IV.

ON IRRATIONAL AND IMAGINARY QUANTITIES.*

(83.) AN IRRATIONAL QUANTITY, or SURD, as it is sometimes called, is a quantity affected by a radical sign, or a fractional index, without which it cannot be accurately expressed; the quantity itself not being susceptible of the extraction which the index denotes.

Thus $\sqrt{2}$ is a surd, because, as 2 is not a square, its square root cannot be accurately extracted; also, $\sqrt{6}$, $3^{\frac{1}{4}}$, $6^{\frac{1}{3}}$, &c. are surds, since none of them are susceptible of the requisite extraction, and therefore cannot be otherwise accurately expressed; although by extending the decimals, in the arithmetical process for extracting the root, we may arrive at a number, which, when raised to the requisite power, will approach to the proposed number within any assigned degree of nearness.

THEOREM 1. The square root of any finite number cannot be partly rational and partly a quadratic surd.

For if $\sqrt{a} = b + \sqrt{c}$, then, by squaring each side, we shall have $a = b^2 + 2b\sqrt{c} + c$, and $\therefore 2b\sqrt{c} = a - b^2 - c$; and, consequently, $\sqrt{c} = \dfrac{a - b^2 - c}{2b}$: that is, an irrational quantity equal to a rational quantity, which is impossible.

As the reasoning holds whether $\sqrt{c}$ be positive or negative, it

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* In a strictly methodical arrangement of the several parts of the Elements of Algebra, the present is the proper place to consider the doctrine of *Surds*; because the expressions so named are of frequent occurrence in the analysis of equations beyond the first degree, and it is desirable that the student, upon entering on such equations, should be prepared to apply to surds, whenever they occur, the reductions necessary to the simplicity and elegance of his results. Nevertheless, the consideration of these reductions may be postponed till some familiarity with *Quadratic Equations*, as taught in next chapter, is acquired.

follows that $\sqrt{a} \pm \sqrt{c}$ cannot be rational. Hence, in a table of the approximate square roots of irrational numbers, however far the approximations be carried, it can never happen that the interminable decimals omitted can in any two cases be the same; for if it could, the difference between these two irrational numbers would be rational.

THEOREM 2. In any equation of the form $a \pm \sqrt{b} = x \pm \sqrt{y}$, consisting of rational quantities and quadratic surds, the rational quantities on each side are equal, as also the irrational quantities.

For if a, x had any difference, as p, then, by transposing, we should have $\sqrt{y} \pm \sqrt{b}$ equal to the rational quantity p, which we have just shown to be impossible. Hence $a = x$, and, consequently, $\sqrt{b} = \sqrt{y}$.

THEOREM 3. If $\sqrt{(a + \sqrt{b})} = x + y$, $\sqrt{b}$ being irrational, then will $\sqrt{(a - \sqrt{b})} = x - y$; x and y being supposed to be one or both quadratic surds.

For since $a + \sqrt{b} = x^2 + 2xy + y^2$, and since x and y are one or both quadratic surds, $x^2 + y^2$ must be rational, and therefore $2xy$ irrational; otherwise, by transposing a, we shall have $\sqrt{b}$ equal to a rational quantity; $\therefore$ (Theor. 2), $a = x^2 + y^2$, and $\sqrt{b} = 2xy$, consequently, $a - \sqrt{b} = x^2 + y^2 - 2xy = (x - y)^2$; $\therefore \sqrt{(a - \sqrt{b})} = x - y$.*

REDUCTION OF SURDS.

(84.) PROBLEM I. *To Reduce a Rational Quantity to the Form of a Surd.*

Raise the quantity to the power denoted by the root of the surd proposed; then the corresponding root of this power, expressed by means of the radical sign, or a fractional index, will be the given quantity under the proposed form.

* This last theorem is employed in investigating the formula for extracting the square root of a binomial surd at p. 114; the others are useful in several analytical inquiries. It is necessary to remark that they are also true when the quantities affected with the radical sign are *negative;* that is, when for *irrational* we substitute *imaginary.* (See p. 119.)

EXAMPLES.

1. Reduce 2 to the form of the square root.
   Here $2^2=4$; $\therefore 2=\sqrt{4}$, the proposed form.
2. Reduce $3x^2$ to the form of the cube root.
   Here $(3x^2)^3=27x^6$; $\therefore 3x^2=\sqrt[3]{27x^6}$.
3. Reduce a^2x^3 to the form of the fifth root.
4. Reduce $\frac{x^2}{y^3}$ to the form of the fourth root.
5. Reduce $\frac{\sqrt{a}}{x}$ to the form of the cube root.
6. Reduce $a^{\frac{1}{3}}x^{\frac{1}{4}}$ to the form of the square root.

Problem II. *To Reduce Surds expressing different Roots to equivalent ones expressing the same Root.*

Bring the indices to a common denominator; then raise each quantity to the power denoted by the numerator of its index, and the common denominator will denote the root of each.

EXAMPLES.

1. Reduce $\sqrt{2}$ and $\sqrt[3]{4}$ to surds expressing the same root.

Here the indices, brought to a common denominator, are $\frac{3}{6}$ and $\frac{2}{6}$;
$\therefore$ the proposed quantities are the same as $2^{\frac{3}{6}}$ and $4^{\frac{2}{6}}$; or $\sqrt[6]{8}$ and $\sqrt[6]{16}$.

2. Reduce $a^{\frac{1}{2}}$ and $a^{\frac{2}{3}}$ to surds expressing the same root.

Here the indices, brought to a common denominator, are $\frac{3}{6}$ and $\frac{4}{6}$;
$\therefore$ the proposed quantities are equivalent to $a^{\frac{3}{6}}$ and $a^{\frac{4}{6}}$; or to $\sqrt[6]{a^3}$ and $\sqrt[6]{a^4}$.

3. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$ to surds expressing the same root.
4. Reduce $2\sqrt[3]{3}$ and $3\sqrt{2}$ to surds expressing the same root.
5. Reduce $6^{\frac{2}{3}}$ and $5^{\frac{3}{4}}$ to surds expressing the same root.
6. Reduce $x^{\frac{3}{4}}$ and $y^{\frac{1}{3}}$ to surds expressing the same root.

PROBLEM III. *To Reduce Surds to their most Simple Forms.*

Surds that admit of simplification are those which may always be divided into two factors, one of which will contain a perfect power corresponding to the surd root.

Hence, to simplify such surds, extract the root of that factor which is the perfect power, and multiply this root by the other factor, with the proper radical sign prefixed.

EXAMPLES.

1. Reduce $\sqrt{a^2b}$ to its most simple form.

   Here, since a^2 is a perfect square, $\sqrt{a^2b} = a\sqrt{b}$.

2. Reduce $\sqrt[3]{135}$ to its most simple form.

   $\sqrt[3]{135} = \sqrt[3]{27 \times 5} = 3\sqrt[3]{5}$, the answer.

3. Reduce $5\sqrt{54}$ to its most simple form.

   $5\sqrt{54} = 5\sqrt{9 \times 6} = 5 \times 3\sqrt{6} = 15\sqrt{6}$, the form required.

4. Reduce $3\sqrt[3]{108}$ to its most simple form.

5. Reduce $\sqrt[3]{ax^3 + bx^6}$ to its most simple form.

6. Reduce $\sqrt[3]{5(a^3 + a^4b)}$ to its most simple form.

(85.) If the surd be in the form of a fraction, it may be decomposed as required, by first multiplying both numerator and denominator by some quantity that will make the denominator of the requisite power. It is plain that instead of the denominator the numerator might be rendered rational: but the resulting form would be less convenient for actual numerical computation, inasmuch as it would involve *division* by the approximate decimal value of the denominator, into which the irrational term would still enter. On this account it is generally considered as inelegant to leave the final result of an algebraic problem affected with an irrational term in the *denominator*, whenever that denominator can be conveniently made rational.

EXAMPLES.

1. Reduce $\sqrt{\frac{3}{8}}$ to its most simple form.

   Here $\sqrt{\frac{3}{8}} = \sqrt{\frac{6}{16}} = \sqrt{\frac{1}{16} \times 6} = \frac{1}{4}\sqrt{6}$.

2. Reduce $\frac{1}{2}\sqrt{\frac{3}{7}}$ to its most simple form.

$$\frac{1}{2}\sqrt{\frac{3}{7}} = \frac{1}{2}\sqrt{\frac{21}{49}} = \frac{1}{2}\sqrt{\frac{1}{49} \times 21} = \frac{1}{14}\sqrt{21}.$$

3. Convert $\dfrac{a}{a + 2\sqrt{a}}$ into a more convenient form.

$$\frac{a}{a + 2\sqrt{a}} = \frac{a\,(a - 2\sqrt{a})}{a^2 - 4a} = \frac{a - 2\sqrt{a}}{a - 4}$$

4. Reduce $\dfrac{a}{b}\sqrt{\dfrac{c^3}{d}}$ to its most simple form.

5. Reduce $5\sqrt[3]{\frac{2}{3}}$ to its most simple form.

6. Reduce $\frac{1}{3}\sqrt[3]{\frac{5}{8}}$ to its most simple form.

7. Reduce $\sqrt{\dfrac{ab^2}{4\,(a + x)}}$ to its most simple form.

8. Reduce $\sqrt{\dfrac{b - 2a}{b}}$ to its most simple form.

9. Reduce $\sqrt{\dfrac{a + b}{a - b}}$ to its most simple form.

ADDITION AND SUBTRACTION OF SURDS.

(86.) Reduce the surds to their most simple forms; then, if the surd part be the same in both, add or subtract the rational parts,* and annex the common surd part to the result; but if the surd parts be different, then the addition or subtraction can only be represented by the proper signs, + or —.

* The rational part is sometimes called the *coefficient* of the surd.

EXAMPLES.

1. What is the sum of $\sqrt{18}$ and $\sqrt{8}$?

$$\left.\begin{array}{l}\text{Here } \sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2} \\ \text{and } \sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}\end{array}\right\}, \therefore 5\sqrt{2} = \text{sum.}$$

2. What is the difference between $\sqrt{108ax^2}$ and $\sqrt{48ax^2}$?

$$\left.\begin{array}{l}\sqrt{108ax^2} = \sqrt{36x^2 \times 3a} = 6x\sqrt{3a} \\ \text{and } \sqrt{48ax^2} = \sqrt{16x^2 \times 3a} = 4x\sqrt{3a}\end{array}\right\}, \therefore 2x\sqrt{3a} = \text{difference.}$$

3. What is the sum of $3\sqrt[3]{32}$ and $2\sqrt[3]{54}$?

$$\left.\begin{array}{l}3\sqrt[3]{32} = 3\sqrt[3]{8 \times 4} = 6\sqrt[3]{4} \\ \text{and } 2\sqrt[3]{54} = 2\sqrt[3]{27 \times 2} = 6\sqrt[3]{2}\end{array}\right\}, \therefore 6\sqrt[3]{4} + 6\sqrt[3]{2} = \text{sum.}$$

4. What is the difference between $\sqrt[3]{192}$ and $\sqrt[3]{24}$?

Ans. $2\sqrt[3]{3}$.

5. What is the sum of $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$?

Ans. $\frac{4}{5}\sqrt{10}$.

6. What is the difference between $\sqrt{\frac{8}{27}}$ and $\sqrt{\frac{1}{6}}$?

Ans. $\frac{1}{18}\sqrt{6}$.

7. What is the sum of $\sqrt{24}$, $2\sqrt{72}$, and $a\sqrt{bx^2}$?

Ans. $2\sqrt{6} + 12\sqrt{2} + ax\sqrt{b}$.

8. Required the sum of $\sqrt[3]{500}$ and $\sqrt[3]{108}$.

Ans. $8\sqrt[3]{4}$.

9. Required the difference between $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$.

Ans. $\frac{2}{5}\sqrt{10}$.

10. Required the difference between $\frac{3}{4}\sqrt{\frac{2}{3}}$ and $\frac{2}{3}\sqrt{\frac{1}{6}}$.

Ans. $\frac{11}{36}\sqrt{6}$.

11. What is the difference between $5\sqrt{20}$ and $3\sqrt{45}$.

Ans. $\sqrt{5}$.

12. What is the sum of $\sqrt{27}$, $\sqrt{48}$, $4\sqrt{147}$, and $3\sqrt{75}$.

Ans. $50\sqrt{3}$.

MULTIPLICATION AND DIVISION OF SURDS.

(87.) Reduce the surds to equivalent ones expressing the same root (Prob. 2), and then multiply or divide as required.

EXAMPLES.

1. Multiply $\sqrt{8}$ by $\sqrt[3]{16}$.

Here $8^{\frac{1}{2}} \times (16)^{\frac{1}{3}} = 8^{\frac{3}{6}} \times (16)^{\frac{2}{6}} = \sqrt[6]{8^3 \times (16)^2} = \sqrt[6]{512 \times 256}$

$= \sqrt[6]{8\,(2)^6 \times 4\,(2)^6} = \sqrt[6]{32\,(4)^6} = 4\sqrt[6]{32} =$ product.

2. Divide $\sqrt{12}$ by $\sqrt[3]{24}$.

Here $\frac{(12)^{\frac{1}{2}}}{(24)^{\frac{1}{3}}} = \frac{(12)^{\frac{3}{6}}}{(24)^{\frac{2}{6}}} = \sqrt[6]{\frac{(12)^3}{(24)^2}} = \sqrt[6]{\frac{27\,(2)^6}{9\,(2)^6}} = \sqrt[6]{3} =$ quotient.

3. Multiply $2\sqrt[3]{\frac{2}{3}}$ by $\sqrt[3]{\frac{5}{8}}$.

Ans. $2\sqrt[3]{15}$.

4. Divide $4\sqrt[3]{ax}$ by $3\sqrt{xy}$.

Ans. $\frac{4\sqrt[6]{a^2x^5y^3}}{3xy}$.

5. Multiply $4\sqrt{3}$ by $3\sqrt[3]{4}$.

Ans. $12\sqrt[6]{432}$.

6. Divide $4\sqrt[6]{32}$ by $\sqrt[3]{16}$.

Ans. $2\sqrt{2}$.

7. Multiply $5a^{\frac{1}{2}}$ by $3a^{\frac{1}{3}}$.

Ans. $15\sqrt[6]{a^5}$.

8. Multiply $2\sqrt{27}$ by $\sqrt{3}$.

Ans. 18.

9. Divide $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$.

Ans. $\frac{3}{4}\sqrt{10}$.

*To Extract the Square Root of a Binomial Surd.**

(88.) Sometimes an algebraic investigation will lead us to a surd result indicating the square root of a quantity, partly rational and partly irrational; that is to say, to the square root of the binomial surd $a \pm \sqrt{b}$. In certain cases such a result may be materially simplified, as it may be reduced either to the form $a' \pm \sqrt{b'}$ or $\sqrt{a'} \pm \sqrt{b'}$, requiring the extraction of the square root of but one, or at most of two quantities, instead of the square root of a square root. As remarked at (85), it is always desirable to effect such reductions when possible, because our final results should be represented in a form the best suited to numerical computation. By means of a table of square roots, the roots of the finite numbers a' and b' might be easily found, whereas the square root of the interminable decimal $a \pm \sqrt{b}$ could not be determined, to any degree of nicety, without numerical labour.

(89.) In order to extract the square root of $a + \sqrt{b}$, put $\sqrt{(a + \sqrt{b})} = x + y$; and it follows that $\sqrt{(a - \sqrt{b})} = x - y$. (Art. 83, Theo. 3.)

Let each of these equations be squared, and we have

$$a + \sqrt{b} = x^2 + 2xy + y^2$$
$$a - \sqrt{b} = x^2 - 2xy + y^2;$$

$\therefore$ by addition . . $2a = 2x^2 + 2y^2$, or $a = x^2 + y^2$.

Let the same two equations be now multiplied together, and there results

$$\sqrt{(a + \sqrt{b})} \times \sqrt{(a - \sqrt{b})} = x^2 - y^2, \text{ or } \sqrt{(a^2 - b)} = x^2 - y^2;$$

hence, both the sum and difference of x^2 and y^2 being given, we have by addition and subtraction,

$$x^2 = \frac{a + \sqrt{(a^2 - b)}}{2}, \text{and } y^2 = \frac{a - \sqrt{(a^2 - b)}}{2};$$

* The term *binomial* is often confined solely to surds of the form $a + \sqrt{b}$, or $\sqrt{a} + \sqrt{b}$; and those of the form $a - \sqrt{b}$, or $\sqrt{a} - \sqrt{b}$, are called *residual* surds.

$$\therefore x = \sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}}, \text{ and } y = \sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}};$$

consequently,

$$\sqrt{(a+\sqrt{b})} = \sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}} + \sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}}$$

$$\sqrt{(a-\sqrt{b})} = \sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}} - \sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}}.$$

(90.) From these forms it appears that in order to the simplification to which we alluded in article (88), the number a^2-b must be a perfect square; in which case, each of the above values will consist either of two surds, or of a rational part and a surd. In all other cases the expression $\sqrt{(a \pm \sqrt{b})}$ is irreducible to a simpler form.

The above formulæ will apply to any particular example, by substituting the particular values for a and b; observing that if b be negative, the signs of b in the formulæ are to be changed.*

EXAMPLES.

1. What is the square root of $8+\sqrt{39}$?

Here $a=8$, and $b=39$;

$$\therefore \sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}} = \sqrt{\left\{\frac{8+\sqrt{(8^2-39)}}{2}\right\}} = \sqrt{\tfrac{13}{2}};$$

$$\text{and } \sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}} = \sqrt{\left\{\frac{8-\sqrt{(8^2-39)}}{2}\right\}} = \sqrt{\tfrac{3}{2}};$$

$$\therefore \sqrt{(8+\sqrt{39})} = \sqrt{\tfrac{13}{2}} + \sqrt{\tfrac{3}{2}} = \tfrac{1}{2}(\sqrt{26}+\sqrt{6}).$$

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* It is well worthy of notice, that in this case of b negative, the quantity inclosed within the second pair of braces, in each of the above expressions, is necessarily *negative*; and that within the first pair necessarily *positive*; so that generally the square root of $a \pm \sqrt{-b}$ is always of the same form, viz. $a' \pm \sqrt{-b'}$; a form which indeed always recurs whatever root or power of it be taken. (See the chapter on the Binomial Theorem.)

2. What is the square root of $10-\sqrt{96}$?

Here $a=10$ and $b=96$;

$$\therefore \sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}}=\sqrt{\left\{\frac{10+\sqrt{(10^2-96)}}{2}\right\}}=\sqrt{6};$$

$$\text{and } \sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}}=\sqrt{\left\{\frac{10-\sqrt{(10^2-96)}}{2}\right\}}=2;$$

$$\therefore \sqrt{(10-\sqrt{96})}=\sqrt{6}-2.$$

3. What is the square root of $6+\sqrt{20}$?

Ans. $1+\sqrt{5}$.

4. What is the square root of $6-2\sqrt{5}$?

Ans. $\sqrt{5}-1$.

5. What is the square root of $7-2\sqrt{10}$?

Ans. $\sqrt{5}-\sqrt{2}$.

6. What is the square root of $42+3\sqrt{174\frac{2}{9}}$?

Ans. $2\sqrt{7}+\sqrt{14}$.

To find Multipliers which will make Binomial Surds rational.

(91.) We have already seen (85) that surds may often be rendered rational by being multiplied by some other quantity, which quantity, when the surd consists of but a simple term, is always easily found: if, for instance, the surd $\sqrt{a}$ is to be freed from its irrational form, it must evidently be multiplied by $\sqrt{a}$; for $\sqrt{a}\times\sqrt{a}=a$; and if $\sqrt[3]{a}$ be the form of the surd, then the multiplier must be $\sqrt[3]{a^2}$; because, $\sqrt[3]{a}\times\sqrt[3]{a^2}=\sqrt[3]{a^3}=a$: and generally, the multiplier that will make $\sqrt[n]{a}$ rational is $\sqrt[n]{a^{n-1}}$; because $\sqrt[n]{a}\times\sqrt[n]{a^{n-1}}=\sqrt[n]{a^n}=a$. The more usual *binomial* forms too may be readily rationalised. Such forms consist of either the sum or difference of two *square roots*, or else of the sum or difference of two *cube roots*. In the former case the multiplier will be suggested from the property, that the product of the sum and difference of two quantities is the difference of their squares. In the latter case the multiplier will be a *trinomial surd*, consisting of the squares of the two given terms, and of their product with its sign changed; that is to say, the form $\sqrt[3]{a}\pm\sqrt[3]{b}$ will be rendered rational by the multiplier $\sqrt[3]{a^2}\mp\sqrt[3]{ab}+\sqrt[3]{b^2}$, since it is

plain that the extreme terms of the product will be rational, and that the four intermediate terms destroy each other. But it is not so easy to discover, at once, the multiplier that will render *any binomial* surd rational; the method of proceeding, however, in this case, is derived from the following investigation:

$$\text{By division}\left\{\begin{array}{l} \dfrac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 + \&c. \\ \dfrac{x^n - y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \&c. \\ \dfrac{x^n + y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \&c. \end{array}\right.$$

Here the first of these series will terminate at the nth term, whether n be even or odd; the second will terminate at the nth term, only when n is an even number; and the third, only when n is an odd number; for, in other cases, they will go on to infinity.*

Now, put $x^n = a$, $y^n = b$; then $x = \sqrt[n]{a}$, and $y = \sqrt[n]{b}$; and the above fractions become, respectively,

$$\frac{a - b}{\sqrt[n]{a} - \sqrt[n]{b}}, \quad \frac{a - b}{\sqrt[n]{a} + \sqrt[n]{b}}, \text{ and } \frac{a + b}{\sqrt[n]{a} + \sqrt[n]{b}};$$

and since $x = \sqrt[n]{a}$, $x^{n-1} = \sqrt[n]{a^{n-1}}$; $x^{n-2} = \sqrt[n]{a^{n-2}}$, &c.;

$$\text{also } y^2 = \sqrt[n]{b^2};\ y^3 = \sqrt[n]{b^3}, \&c.;$$

and, substituting these values in the above quotients,

$$\text{we have}\left\{\begin{array}{l} \dfrac{a - b}{\sqrt[n]{a} - \sqrt[n]{b}} = \sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} + \sqrt[n]{a^{n-4}b^3} + \&c. \\ \dfrac{a - b}{\sqrt[n]{a} + \sqrt[n]{b}} = \sqrt[n]{a^{n-1}} - \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} - \sqrt[n]{a^{n-4}b^3} + \&c. \\ \dfrac{a + b}{\sqrt[n]{a} + \sqrt[n]{b}} = \sqrt[n]{a^{n-1}} - \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} - \sqrt[n]{a^{n-4}b^3} + \&c. \end{array}\right.$$

therefore, since the divisor multiplied by the quotient produces the

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* Each of the above series is a geometrical series, the common ratio in the first being $\frac{y}{x}$, and in the others $-\frac{y}{x}$; and, by the formula at p. 99, the sums of these, *as far as n terms*, will be the fractions which have generated them only in the cases stated above.

dividend, it follows, that if a surd of the form $\sqrt[n]{a} - \sqrt[n]{b}$, be multiplied by

$$\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} + \&c.$$

to n terms, the product will be $a - b$, a rational quantity; also, if a surd of the form $\sqrt[n]{a} + \sqrt[n]{b}$, be multiplied by $\sqrt[n]{a^{n-1}} - \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} - \&c.$ to n terms, the product will be $a - b$, or $a + b$, according as n is even or odd, each of which products is a rational quantity.

The Examples which follow, although worked by the general formulas here given, may nevertheless be readily solved by the two simple rules stated at the commencement of this article.

EXAMPLES.

1. It is required to find a multiplier that shall make $\sqrt[3]{7} - \sqrt[3]{5}$ rational.

Here $a = 7$, $b = 5$, and $n = 3$; $\therefore$ the multiplier, $\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}b} + \&c.$ is in this case $\sqrt[3]{49} + \sqrt[3]{35} + \sqrt[3]{25}$,

for by multiplication $\sqrt[3]{7} - \sqrt[3]{5}$

$$\begin{array}{llll}
\sqrt[3]{343} & + \sqrt[3]{245} & + \sqrt[3]{175}, & \\
 & - \sqrt[3]{245} & - \sqrt[3]{175} & - \sqrt[3]{125} \\
\hline
\sqrt[3]{343} & \quad * & \quad * & - \sqrt[3]{125} = 7 - 5.
\end{array}$$

2. It is required to find a multiplier that will make $2 + \sqrt[3]{2}$ rational.

Here $2 + \sqrt[3]{3} = \sqrt[3]{8} + \sqrt[3]{3}$, $\therefore a = 8$, $b = 3$, and $n = 3$; $\therefore$ the multiplier, $\sqrt[n]{a^{n-1}} - \sqrt[n]{a^{n-2}b} + \&c.$

$$= \sqrt[3]{64} - \sqrt[3]{24} + \sqrt[3]{9} = 4 - 2\sqrt[3]{3} + \sqrt[3]{9}.$$

3. It is required to convert $\dfrac{3}{\sqrt[3]{5} - \sqrt[3]{2}}$ into a fraction that shall have a rational denominator.

$$\text{Ans. } \frac{3(\sqrt[3]{25} + \sqrt[3]{10} + \sqrt[3]{4})}{5 - 2} = \sqrt[3]{25} + \sqrt[3]{10} + \sqrt[3]{4}.$$

4. It is required to convert $\dfrac{a}{\sqrt{a} + \sqrt{b}}$ into a fraction that shall have a rational denominator.

$$\text{Ans. } \frac{a(\sqrt{a} - \sqrt{b})}{a - b}.$$

5. It is required to convert $\frac{a}{\sqrt[3]{x}+\sqrt[3]{y}}$ into a fraction that shall have a rational denominator.

Ans. $\frac{a(\sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2})}{x+y}$.

6. It is required to find a multiplier that will make $\sqrt[4]{3}+\sqrt[4]{4}$ rational.

Ans. $\sqrt[4]{27}-\sqrt[4]{36}+2\sqrt[4]{3}-2\sqrt[4]{4}$.

The answer here given to this last example has, like those above, been determined from the general formulas; but the proper multiplier may be more readily obtained, and in a preferable form, by the rule at the commencement of the article: thus the multiplier $\sqrt[4]{3}-\sqrt[4]{4}$ will bring the proposed form to $\sqrt{3}-\sqrt{4}$, and the multiplier $\sqrt{3}+\sqrt{4}$ reduces this last to $3-4$, so that the complete multiplier sought consists of the two factors $\sqrt[4]{3}-\sqrt[4]{4}$ and $\sqrt{3}+2$, which produce the multiplier in the answer.

ON IMAGINARY QUANTITIES.

(92.) Imaginary quantities are those expressions which represent any *even* root of a negative quantity, as $\sqrt{-a}$, $\sqrt[4]{-a}$, &c. the values of such expressions being unassignable, because from the nature of involution no such even root can exist. These quantities differ from other surd expressions, inasmuch as the values of the latter, though inexpressible accurately, may still be approximated to; but imaginaries are not susceptible even of approximate values, being the mere symbols of *absurdity*, to which we are led in the solution of problems of the second and superior degrees whenever the conditions of such problems are contradictory or absurd; notwithstanding this, however, they are of considerable use in various parts of the mathematics, and when subjected to the ordinary rules of algebraic calculation, often lead to interesting and valuable results.

(93.) With respect to the addition and subtraction of these quantities, the operations are the same as for algebraic expressions in general; but, as regards their multiplication and division, several particulars must be attended to that do not attach to other quantities; and which we shall here enter upon.

(94.) It is evident, in the first place, that $\sqrt{-a} \times \sqrt{-a}$ must be equal to $-a$; for the square root of any quantity multiplied by that square root must produce the original quantity, and therefore no ambiguity can here arise with respect to the sign of a. It is also equally evident that $\sqrt{-a} \times \sqrt{-a}$ must be equal to $\sqrt{a^2}$; for if this were not the case, the rule for the signs in multiplication could not be general; it therefore follows, that $-a$ must be equal to $\sqrt{a^2}$.

But it may be said that $\sqrt{a^2}$ is also $=a$, and that therefore it would follow that $a=-a$; this reasoning is, however, erroneous; for it is not true that $\sqrt{a^2}$ is *also* $= a$, since the result of the operation implied in the symbol $\sqrt{}$ does not take *both* the signs $+$ and $-$, but *either* $+$ or $-$; and, consequently, if it be shown to take the one, it cannot at the same time *also* have the other; in the present case, therefore, the root takes only the *minus* sign; and, consequently,

$$\sqrt{-a} \times \sqrt{-a} = -\sqrt{a^2} = -a.$$

(95.) Our being able to destroy the ambiguity of the symbol $\sqrt{}$ in the expression $\sqrt{a^2}$, arose solely from our previous knowledge of the manner in which a^2 was produced, viz. from the involution of $-a$; and that therefore the reverse operation, being performed on a^2, must bring back the original quantity $-a$. If we had had no knowledge of the generation of a^2, whether it was produced from $(+a) \times (+a)$, or from $(-a) \times (-a)$; that is, whether a^2 represented $(+a)^2$, or $(-a)^2$; then, in the reverse operation, we could of course have had no knowledge of the precise quantity which ought to have been produced; that is, the symbol of extraction would have been ambiguous, and the operation could only have been expressed by saying $\sqrt{a^2} = +a$, or $-a$.

In the same manner, if it be known that a^2 is produced from $(+a) \times (+a)$, then $\sqrt{a^2} = +\sqrt{a^2} = +a$.

(96.) Again, if we have two *unlike* imaginary quantities, $\sqrt{-a}$ and $\sqrt{-b}$, we know that their product, $\sqrt{-a} \times \sqrt{-b} = \sqrt{ab}$; but as in this case the quantity, ab, whose root is to be extracted, is not generated from that root, but from two unequal factors, when the numerical value of the root is actually determined, we shall find ourselves unprovided with any reason why one of the two signs should be prefixed to it any more than the other. The ambiguity here is similar to that which belongs to the ordinary case of real quantities. The product $\sqrt{+a} \times \sqrt{+b}$ equally gives $\sqrt{ab}$, and equally leaves the sign of the root undetermined. Nevertheless, the two results are not identical; for although the same in numerical value, and equally ambiguous, yet, *whenever the one is taken plus the other must be taken minus, and vice versâ*. This will be seen by decomposing the proposed factors; for since

$\sqrt{-a} = \sqrt{a} \times \sqrt{-1}$, and $\sqrt{-b} = \sqrt{b} \times \sqrt{-1}$, we have

$$\sqrt{-a} \times \sqrt{-b} = (\sqrt{a} \times \sqrt{-1})(\sqrt{b} \times \sqrt{-1})$$
$$= \sqrt{ab} \times -1 = -\sqrt{ab}.$$

Hence it appears that whatever sign be attributed to $\sqrt{ab}$, when produced from two *real* factors, the opposite sign takes its place when the producing factors are *imaginary*; so that if in the former case we write the ambiguity thus, $\pm \sqrt{ab}$, then in the latter case it must be written $\mp \sqrt{ab}$.

By the decomposition employed above may any imaginary be represented by two factors, of which one is a real quantity and the other the imaginary $\sqrt{-1}$; and therefore the expression $\sqrt{-1}$ may be considered as a universal factor of every imaginary quantity, the other factor being a real quantity, either rational or irrational.

(97.) From what has been just said, and from the property that the multiplication of like signs always produces *plus*, it follows that

The product of two imaginaries that have the same sign is equal to *minus* the square root of their product, considering them as real quantities. That is,

$$(+\sqrt{-a})\,(+\sqrt{-a})=-\sqrt{a^2}=-a;$$

as also

$$(-\sqrt{-a})\,(-\sqrt{-a})=-\sqrt{a^2}=-a;$$

and

$$(+\sqrt{-a})\,(+\sqrt{-b})=-\sqrt{ab};$$

as also

$$(-\sqrt{-a})\,(-\sqrt{-b})=-\sqrt{ab}.$$

(98.) But if the two imaginaries have different signs, then their product will evidently be equal to *plus* the square root of their product, considering them as real. That is,

$$(+\sqrt{-a})\,(-\sqrt{-b})=+\sqrt{ab},$$

the resulting sign being always opposite to that which would have place if the factors were real, as shown above.

EXAMPLES.

1. Multiply $2\sqrt{-3}$ by $3\sqrt{-2}$.

   Here $2\sqrt{-3}\times 3\sqrt{-2}=-6\sqrt{6}$.

2. Multiply $3+\sqrt{-2}$ by $2-\sqrt{-4}$.

$$\begin{array}{l} 3+\sqrt{-2} \\ 2-\sqrt{-4} \\ \hline 6+2\sqrt{-2} \\ \quad -3\sqrt{-4}+\sqrt{8} \\ \hline 6+2\sqrt{-2}-3\sqrt{-4}+\sqrt{8} \\ \hline \end{array}$$

3. Multiply $4\sqrt{-5}$ by $3\sqrt{-1}$.

   Ans. $-12\sqrt{5}$.

4. Multiply $-5\sqrt{-2}$ by $-3\sqrt{-5}$.

   Ans. $-15\sqrt{10}$.

5. Multiply $4+\sqrt{-3}$ by $\sqrt{-5}$.

   Ans. $4\sqrt{-5}-\sqrt{15}$.

6. Required the cube of $a-b\sqrt{-1}$.

   Ans. $a^3+b^3\sqrt{-1}-3ab\,(b+a\sqrt{-1})$.

   or $a^3-3ab^2+b\,(b^2-3a^2)\sqrt{-1}$.

It may be noticed, in connexion with the present article, that as the decomposition of the *difference* of two squares into *real* factors is often found useful in the earlier operations of algebra, so the decomposition of the *sum* of two squares into *imaginary* factors, is in frequent request in the more advanced departments of analysis. This decomposition is obviously

$$x^2+y^2=(x+\sqrt{-y})\,(x-\sqrt{-y}).$$

(99.) The quotient of two imaginaries having the same sign is equal to *plus* the square root of their quotient, considering them as real quantities. That is,

$$\frac{+\sqrt{-a}}{+\sqrt{-b}}=\frac{+\sqrt{a}\times\sqrt{-1}}{+\sqrt{b}\times\sqrt{-1}}=+\sqrt{\frac{a}{b}};$$

as also

$$\frac{-\sqrt{-a}}{-\sqrt{-b}}=\frac{-\sqrt{a}\times\sqrt{-1}}{-\sqrt{b}\times\sqrt{-1}}=+\sqrt{\frac{a}{b}}.$$

(100.) But if the two imaginaries have different signs, it is evident that their quotient will be equal to *minus* the square root of their quotient, considering them as real quantities.

EXAMPLES.

1. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$.

$$\frac{6\sqrt{-3}}{2\sqrt{-4}}=3\sqrt{\tfrac{3}{4}}=\tfrac{3}{2}\sqrt{3}.$$

2. Divide $1+\sqrt{-1}$ by $1-\sqrt{-1}$.

Here the multiplier that will render $1-\sqrt{-1}$ rational is $1+\sqrt{-1}$ (Art. 91).

$$\therefore\frac{1+\sqrt{-1}}{1-\sqrt{-1}}=\frac{2\sqrt{-1}}{2}=\sqrt{-1}.$$

2. Divide $2\sqrt{-7}$ by $-3\sqrt{-5}$.

Ans. $-\tfrac{2}{3}\sqrt{\tfrac{7}{5}}=-\tfrac{2}{15}\sqrt{35}$.

4. Divide $-\sqrt{-1}$ by $-6\sqrt{-3}$.

Ans. $+\dfrac{1}{6\sqrt{3}}=\dfrac{\sqrt{3}}{18}$.

5. Divide $4+\sqrt{-2}$ by $2-\sqrt{-2}$.

Ans. $1+\sqrt{-2}$.

6. Divide $3+2\sqrt{-1}$ by $3-2\sqrt{-1}$.

Ans. $\frac{1}{13}(5+12\sqrt{-1})$.

SCHOLIUM.

(101.) It was observed at the commencement of this subject that imaginary quantities always occur in the analysis of a problem of the second degree, when its conditions involve any absurdity or impossibility; as if it were proposed to divide the number 12 into two parts, such, that their product may be 40. If this question be solved by the ordinary rules for the solution of a quadratic equation, the two parts will be found to be $6+\sqrt{-4}$, and $6-\sqrt{-4}$, being both imaginary or impossible in numbers.* But besides this, the use of imaginaries, as we have before said, is very extensive in some of the higher branches of analysis, and their application to a variety of highly interesting particulars has lately been shown by Mr. Benjamin Gompertz, in his Tracts on 'The Principles and Application of Imaginary Quantities.'

* In problems of the *first degree*, impossible conditions will, as we have already seen at page 85, either conduct us to some palpably absurd conclusion, or else to the vague symbol $\frac{0}{0}$, the occurrence of which is sufficient to induce a suspicion that the conditions which have led to it may be incompatible; and this suspicion is to be either removed or confirmed by an actual examination of those conditions.

CHAPTER V.

QUADRATIC EQUATIONS.

QUADRATICS INVOLVING ONLY ONE UNKNOWN QUANTITY.

(102.) A QUADRATIC, as has been already defined, is an equation that contains the *second* but no higher power of the unknown quantity or quantities, which second power can be removed from the equation only by the extraction of the square root.

It sometimes happens that the higher powers of the unknown quantity entering an equation, enter in such a way as to neutralize or destroy one another, and disappear by transposition; in other cases advanced powers may arise from the unknown entering as a factor into all the terms of the equation, which factor may be removed by division. An equation may thus, in appearance, be a quadratic, or one of a higher degree, and yet be reducible to a simple equation by the common preparatory operations of transposition, division, &c. Such an equation, therefore, is to be regarded only as a simple equation in disguise. A quadratic properly so called, is an equation which can be reduced to a simple equation only by the extraction of the square root.

(103.) Quadratic equations, involving but one unknown quantity, are obviously either of the form

$$\pm ax^2 \pm b = \pm c,$$

$$\text{or } \pm ax^2 \pm bx \pm c = \pm d;$$

and accordingly, as they come under the first or second of these

forms, they are said to be PURE QUADRATICS, or ADFECTED QUADRATICS.

(104.) The solution of a Pure Quadratic is obviously a matter of but little difficulty; for, since it contains but one unknown term, $\pm ax^2$, if this term be made to stand by itself on one side of the equation, and the known terms on the other side, then the division of both sides by $\pm a$ will evidently produce an equation expressing the value of x^2; and the square root of this value must give that of x.

We shall therefore proceed to

ADFECTED QUADRATIC EQUATIONS.

(105.) Let $\pm ax^2 \pm bx \pm c = \pm d$ be an adfected quadratic equation; then, by transposing and dividing by $\pm a$, it becomes

$$x^2 \pm \frac{b}{a} x = \frac{\pm d \mp c}{\pm a};$$

or, putting p for $\frac{b}{a}$, and $\pm q$ for $\frac{\pm d \mp c}{\pm a}$, it is

$$x^2 \pm px = \pm q.$$

Add now the square of $\frac{1}{2}p$ to each side of this equation, and there results

$$x^2 \pm px + \tfrac{1}{4}p^2 = \pm q + \tfrac{1}{4}p^2,$$

where it is readily perceived that the first side is a complete square, viz. $(x \pm \frac{1}{2}p)^2$; consequently, if the square root of each side be extracted, we obtain $x \pm \frac{1}{2}p = \pm \sqrt{\pm q + \frac{1}{4}p^2}$ (the double sign $\pm$ being placed before the radical, because the square root of a quantity may be either $+$ or $-$); hence it appears that

$$x = \mp \tfrac{1}{2}p + \sqrt{\pm q + \tfrac{1}{4}p^2} \text{ or } \mp \tfrac{1}{2}p - \sqrt{\pm q + \tfrac{1}{4}p^2},*$$

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* These general expressions for x may also be obtained as follows: Having reduced the proposed equation to the form $x^2 \pm px = \pm q$,

either of which values, being substituted for x in the proposed equation, will satisfy its conditions or render its two members identical. These values are called the *roots* of the equation; and thus every quadratic equation is said to have two roots.

(106.) The above general values of x evidently include every possible case, from which separate formulæ for each distinct case are readily obtained, and are as follow:

In equations of the form

$$x^2+px=q,\ x=\begin{cases} -\frac{1}{2}p+\sqrt{q+\frac{1}{4}p^2}, \\ \text{or } -\frac{1}{2}p-\sqrt{q+\frac{1}{4}p^2}, \end{cases}$$

$$x^2-px=q,\ x=\begin{cases} \frac{1}{2}p+\sqrt{q+\frac{1}{4}p^2}, \\ \text{or } \frac{1}{2}p-\sqrt{q+\frac{1}{4}p^2}, \end{cases}$$

$$x^2+px=-q,\ x=\begin{cases} -\frac{1}{2}p+\sqrt{-q+\frac{1}{4}p^2}, \\ \text{or } -\frac{1}{2}p-\sqrt{-q+\frac{1}{4}p^2}, \end{cases}$$

$$x^2-px=-q,\ x=\begin{cases} \frac{1}{2}p+\sqrt{-q+\frac{1}{4}p^2}, \\ \text{or } \frac{1}{2}p-\sqrt{-q+\frac{1}{4}p^2}, \end{cases}$$

(107.) In the last two forms, if q be greater than $\frac{1}{4}p^2$, then $\sqrt{-q+\frac{1}{4}p^2}$ will be impossible, being the square root of a negative quantity; so that if one value be impossible, the other is impossible also.

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as above, let us proceed actually to extract the square root of the first side. The process is as follows:

$$\begin{array}{r|l} & x^2 \pm px \quad \underline{x \pm \frac{1}{2}p} \\ & x^2 \\ \hline 2x \pm \frac{1}{2}p & \pm px \\ & \pm px + \frac{1}{4}p^2 \end{array}$$

It is obvious, from this, that the proposed expression is not a complete square, being indeed deficient by the quantity $\frac{1}{4}p^2$. If, therefore, we add this quantity to each side of the equation, and then extract the square root, we shall have, as above,

$$x \pm \tfrac{1}{2}p = \pm\sqrt{\pm q + \tfrac{1}{4}p^2}.$$

From the above formulæ* the value of the unknown, in any particular example, may be obtained by substitution; or the operations to be performed may be expressed at length, as follows:

(108.) 1. Bring all the unknown terms to one side of the equation, and the known terms to the other.

2. Divide each side of the equation by the coefficient of the unknown square, if it have a coefficient.

3. Add the square of half the coefficient of the simple unknown to each side of the equation, and the unknown side will then be a complete square.

4. Extract the square root of each side, and from the result the value of the unknown quantity is immediately deducible.

EXAMPLES.

1. Given $x^2+6x+4=59$, to find the values of x.

By transposition, $x^2+6x=55$.

and completing the square, $x^2+6x+9=64$;

$\therefore$ extracting the root, $x+3=\pm\sqrt{64}=\pm 8$;

whence $x=5$, or -11.

So that whether we substitute 5 or -11 for x, in the first member of the proposed equation, the numerical result will in either case be 59.

2. Given $2x^2+12x+36=356$, to find the values of x.

By transposition, $2x^2+12x=320$;

or dividing by 2, $x^2+6x=160$,

and completing the square, $x^2+6x+9=169$;

$\therefore$ extracting the root, $x+3=\pm\sqrt{169}=\pm 13$;

whence $x=10$, or -16.

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* Any general rule, expressed in algebraical language, is called a *formula*.

3. Given $10x^2-8x+6=318$, to find the values of x.

By transposition, $10x^2-8x=312$;

or dividing by 10, $x^2-\frac{4}{5}x=\frac{156}{5}$,

and completing the square, $x^2-\frac{4}{5}x+\frac{4}{25}=\frac{784}{25}$;

$\therefore$ extracting the root, $x-\frac{2}{5}=\pm\sqrt{\frac{784}{25}}=\pm\frac{28}{5}$;

whence $x=6$, or $-5\frac{1}{5}$.

4. Given $4x=\dfrac{36-x}{x}+46$, to find the values of x.

Clearing of fractions, $4x^2=36-x+46x=36+45x$;

and by transposition, $4x^2-45x=36$;

or $x^2-\frac{45}{4}x=9$,

and completing the square, $x^2-\frac{45}{4}x+(\frac{45}{8})^2=9+(\frac{45}{8})^2=\frac{2601}{64}$;

$\therefore$ extracting the root, $x-\frac{45}{8}=\pm\sqrt{\frac{2601}{64}}=\pm\frac{51}{8}$;

whence $x=12$, or $-\frac{3}{4}$.

5. Given $5x-\dfrac{3x-3}{x-3}=2x+\dfrac{3x-6}{2}$, to find the values of x.

Clearing of fractions.

$$10x^2-36x+6=4x^2-12x+3x^2-15x+18;$$

and by transposition, $3x^2-9x=12$;

or $x^2-3x=4$;

and completing the square, $x^2-3x+\frac{9}{4}=4+\frac{9}{4}=\frac{25}{4}$;

$\therefore$ extracting the root, $x-\frac{3}{2}=\pm\sqrt{\frac{25}{4}}=\pm\frac{5}{2}$;

whence $x=4$, or -1.

6. Given $\dfrac{3}{x^2-3x}+\dfrac{6}{2x^2+8x}=\dfrac{27}{8x}$, to find the values of x.

Dividing by $\dfrac{3}{x}$, $\dfrac{1}{x-3}+\dfrac{1}{x+4}=\frac{9}{8}$;

and clearing the equation of fractions,

$$8x+32+8x-24=9x^2+9x-108;$$

$\therefore$ by transposition, $116=9x^2-7x$, or rather $9x^2-7x=116$,

$\therefore x^2-\frac{7}{9}x=\frac{116}{9}$;

and completing the square,

$$x^2-\tfrac{7}{9}x+(\tfrac{7}{18})^2=\tfrac{116}{9}+(\tfrac{7}{18})^2=\tfrac{4225}{324};$$

$\therefore$ extracting the root, $x-\frac{7}{18}=\pm\sqrt{\frac{4225}{324}}=\pm\frac{65}{18}$;

whence $x=4$, or $-\frac{29}{9}$.

7. Given $\sqrt{3x-5}=\dfrac{\sqrt{7x^2+36x}}{x}$, to find the values of x.

Squaring each side, $3x-5=\dfrac{7x^2+36x}{x^2}=\dfrac{7x+36}{x}$;

and, multiplying by x, $3x^2-5x=7x+36$;

or, by transposition, $3x^2-12x=36$;

$$\therefore x^2-4x=12;$$

and completing the square, $x^2-4x+4=16$;

$\therefore$ extracting the root, $x-2=\pm 4$;

$$\text{whence } x=6, \text{ or } -2.$$

8. Given $\sqrt{(4+x)(5-x)}=2x-10$, to find the values of x.

Squaring each side, $20+x-x^2=4x^2-40x+100$;
and by transposition, $5x^2-41x=-80$;

$$\therefore x^2-\tfrac{41}{5}x=-16;$$

and completing the square,

$$x^2-\tfrac{41}{5}x+(\tfrac{41}{10})^2=-16+(\tfrac{41}{10})^2=\tfrac{81}{100};$$

$\therefore$ extracting the root, $x-\frac{41}{10}=\pm\sqrt{\frac{81}{100}}=\pm\frac{9}{10}$;

$$\therefore x=5, \text{ or } \tfrac{32}{10}=3\tfrac{1}{5}.$$

NOTE.* If each of these values of x be substituted in the proposed equation, we shall find that the first satisfies its conditions only when the root in the first member is taken positively, and that the second value satisfies the conditions only when the root is taken negatively. The *two* values, therefore, are each admissible, only on the hypothesis that the question requires us to find such values of x as will cause the expression $2x-10$ to become *a root* (either positive or negative) of $(4+x)(5-x)$. If, however, as the ambiguous sign $\pm$ is not prefixed to the first member, that member be regarded as exclusively *positive*, then the value $x=3\frac{1}{5}$ must be rejected, since it will not satisfy the proposed equation under this restriction. Such restriction, however, is altogether

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* Should the above NOTE prove at all embarrassing to the young student, he is recommended to postpone the consideration of it till he has arrived at the end of the present chapter.

inoperative throughout the process of solution, even from the first step; since, from the nature of involution, $20 + x - x^2$ is the square of the root $\sqrt{(4+x)(5-x)}$, whether this root be $+$, or $-$, or entirely ambiguous. The two values of x both belong to the equation $20 + x - x^2 = 4x^2 - 40x + 100$, which is in reality the *quadratic* equation that has been solved; the restriction, however, introduced into the equation from which this is deduced, is to the effect that only *one* of the roots of the quadratic can be admitted. The solution of the quadratic very properly furnishes us with *both* its roots, so that it afterwards remains for us to ascertain, by actual trial, which of these it is that comes within the restriction, and to reject the other. In the proposed hypothesis, $x = 5$ is the only admissible solution. It must be remembered that when the operation $\sqrt{}$ is *introduced* into an algebraical process, thus becoming a part of that process, the ambiguous sign always accompanies the result, in the absence of all hypothesis respecting the generation of the expression operated upon. But when $\sqrt{}$ is not introduced into the solution, but enters the data, then the sign prefixed to $\sqrt{}$ may or may not be controlled by those data; that is, the data may or may not involve a condition as to the generation of the quantity under $\sqrt{}$.

Some misconception, however, has arisen, with respect to the subject of this note, from what may be regarded as an ambiguity of notation. Thus the expression $(2x - 5) + \sqrt{x^2 - 7}$ has been considered by some as $(2x - 5)$ *plus* a root (either positive or negative) of $x^2 - 7$; and by others as $(2x - 5)$ *together with* the *plus* root of $\sqrt{x^2 - 7}$; in the former case the expression has two values, and in the latter only one of those two. Now it is of importance to observe that in this latter case, whenever the proposed expression is required to fulfil *another* condition, the two conditions may be incompatible, so that no value of x, either real or imaginary, will be competent to satisfy them both. For instance, if the new condition be that

$$(2x - 5) + \sqrt{x^2 - 7} = 0, \ldots [A]$$

then the two conditions are; 1st, that to $2x - 5$ is to be added a

positive number equal to the square root of x^2-7; and 2d, that the result is to be equal to zero.

It is easy to see that these conditions are contradictory. For, by transposing,

$$+\sqrt{x^2-7}=5-2x,$$

that is, $5-2x$ must be *positive;* and consequently x must not exceed $2\frac{1}{2}$. But unless x do exceed $2\frac{1}{2}$, the first member will be imaginary; hence the conditions are incompatible. That no *imaginary* value for x can exist, whatever be the sign of the root, is plain; because we should in that case have an imaginary quantity in the first member equal to a quantity partly real and partly imaginary in the second member; which is impossible (83).

If we proceed with the proposed equation [A] in the usual manner, and clear it of the radical sign, we shall be led to the quadratic

$$3x^2-20x+32=0, \ldots .[B]$$

of which the roots are found to be 4 and $\frac{8}{3}$. Now it should be no cause of surprise to the student that both these values satisfy the equation [B] and yet that neither of them satisfies the equation [A], from which [B] has been derived. For [B] has been derived from [A] by multiplying the first member of [A] by a new factor, the factor $(2x-5)-\sqrt{x^2-7}$; since the operation of clearing the radical sign is really equivalent to such multiplication. Hence, although, as we have just shown, it is impossible to render the first member of [A] zero, yet the first member of [B] will be zero, provided only the new factor, which has multiplied [A], be zero; since for a product to be zero it is obviously sufficient that *either* of its factors be zero. The first member of [B], which is formed from the multiplication of the two factors $(2x-5)+\sqrt{x^2-7}$ and $(2x-5)-\sqrt{x^2-7}$, becomes zero, as we have already seen, for these two values of x, viz. $x=4$, $x=\frac{8}{3}$; and as we have also seen that the first of these factors cannot possibly become zero, it follows that it is the second factor, the new multiplier introduced, that vanishes for these values of x.

The inference to be drawn from these observations is this: that whenever an equation involving radical signs is proposed for solution, and it be a condition that the sign which precedes any irrational term denotes exclusively the character of the root; then, if in preparing such equation for solution, we are led to a quadratic, the roots of this quadratic will not *necessarily* satisfy the proposed: they may one or both belong only to the *foreign factor* virtually introduced in the preparatory operation; so that it will be necessary actually to substitute these roots in the proposed equation in order to ascertain whether they really satisfy its conditions or not.

But whenever the sign which precedes the radical merely implies the algebraical addition or subtraction of *a* root of the quantity under that radical, without any restriction as to the *sign* of the root, then both roots of the resulting quadratic will invariably satisfy the proposed equation. In the examples which follow, the signs which precede the radicals are to be thus understood.

For further particulars on the subject of this note, the inquiring student may consult the *Philosophical Magazine* for January and July, 1836; also the *Gentleman's Diary* for 1837, pp. 34, 35.

1. Given $2x^2+8x-6=4$, to find the values of x.

Ans. $x=1$, or -5.

2. Given $3x^2-18x-21=0$, to find the values of x.

Ans. $x=7$, or -1.

3. Given $x-\frac{2}{x}+6=\frac{14}{x}$, to find the values of x.

Ans. $x=2$, or -8.

4. Given $\frac{x^2-8x}{3}+5=0$, to find the values of x.

Ans. $x=3$, or 5.

5. Given $\frac{x^2+8x}{3}+5=0$, to find the values of x.

Ans. $x=-3$, or -5.

6. Given $5x+4=\frac{3\,(2+x^2)}{x}$, to find the values of x.

Ans. $x=1$, or -3.

7. Given $x^2-5x+6=0$, to find the values of x.

Ans. $x=2$, or 3.

8. Given $x^2+3x=28$, to find the values of x.

Ans. $x=4$, or -7.

9. Given $8x^2+6=7x+171$, to find the values of x.

Ans. $x=5$, or $-4\frac{1}{8}$.

10. Given $3x^2=42-5x$, to find the values of x.

Ans. $x=3$, or $-4\frac{2}{3}$.

11. Given $4x-\frac{36-x}{x}=46$, to find the values of x.

Ans. $x=12$, or $-\frac{3}{4}$.

12. Given $\frac{6\,(2x-11)}{x-3}+x-2=24-3x$, to find the values of x.

Ans. $x=6$, or $\frac{1}{2}$.

13. Given $\frac{120}{3x+1}+\frac{90}{x}=42$, to find the values of x.

Ans. $x=3$, or $-\frac{5}{21}$.

14. Given $x^3+(19-x)^3=1843$, to find the values of x.

Ans. $x=11$, or 8.

15. Given $325+x:x::245+x:60$, to find the values of x.

Ans. $x=75$, or -260.

16. Given $\frac{x}{2}\left(\frac{x^2}{4}-1\right)=\frac{x-2}{4}$, to find the values of x.

Ans. $x=-1\pm\sqrt{3}$.

17. Given $\{34-3(x-1)\}\,\frac{x}{2}=57$, to find the values of x.

Ans. $x=6$, or $6\frac{1}{3}$.

18. Given $\frac{10}{x}-\frac{14-2x}{x^2}=\frac{22}{9}$, to find the values of x.

Ans. $x=3$, or $\frac{21}{11}$.

19. Given $\frac{y^3-10y^2+1}{y^2-6y+9}=y-3$, to find the values of y.

Ans. $y=1$, or -28.

20. Given $\frac{6x^2-23x+10}{9-2x}=-7x+42$, to find the values of x.

Ans. $x=11\frac{1}{2}$, or 4.

21. Given $x+\frac{\sqrt{x-3}}{2}=8$, to find the values of x.

Ans. $x=9\frac{1}{4}$, or 7.

22. Given $2x+\frac{x^3}{\sqrt{2x^4-3x^3}}=2x(x+1)$, to find the values of x.

Ans. $x=\frac{3\pm\sqrt{11}}{4}$.

23. Given $(x+a)\sqrt{\frac{x}{a}}=(x-a)\sqrt{\frac{a}{x}}$, to find the values of x.

Ans. $x=\pm a\sqrt{-1}$.

24. Given $\sqrt{4+\sqrt{2x^3+x^2}}=\frac{x+4}{2}$, to find the values of x.

Ans. $x=12$, or 4.

25. Given $x^{\frac{3}{2}}+x^{\frac{5}{2}}=6x^{\frac{1}{2}}$, to find the values of x.

Ans. $x=2$, or -3.

26. Given $\sqrt[3]{x^3-a^3}=x-b$, to find the values of x.

$$\text{Ans. } x=\frac{b}{2}\pm\sqrt{\frac{4a^3-b^3}{12b}}.$$
$$=\frac{b}{2}\pm\frac{\sqrt{3b(4a^3-b^3)}}{6b}.$$

27. Given $\frac{x+a}{x}+\frac{x}{x+a}=b$, to find the values of x.*

$$\text{Ans. } x=\frac{a}{2}\left\{-1\pm\sqrt{\frac{b+2}{b-2}}\right\}.$$
$$=\frac{a}{2}\left\{-1\pm\frac{\sqrt{(b^2-4)}}{b-2}\right\}.$$

28. Given $\sqrt{a+x}+\sqrt{b+x}=\sqrt{a+b+2x}$, to find the values of x.

Ans. $x=-a$, or $-b$.

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* By putting y for $\frac{x+a}{x}$, this equation will take the more simple form $y+\frac{1}{y}=b$.

29. Given $\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}, =x$, to find the values of x.

Ans. $x=\frac{1\pm\sqrt{5}}{2}$.

30. Given $x-\frac{12+8\sqrt{x}}{x-5}=0$, to find the values of x.

Ans. $x=9$, or 4.

31. Given $x+\sqrt{x} : x-\sqrt{x} :: 3\sqrt{x}+6 : 2\sqrt{x}$, to find the values of x.

Ans. 9 or 4.

32. Given the first term a, the difference d, and the sum s of an arithmetical series, to find the number of terms n.

Ans. $n=\frac{1}{2d}\{d-2a\pm\sqrt{[(2a-d)^2+8ds]}\}$.

If the terms of the arithmetical series be written in reverse order, l will take the place of a, and the sign of d will become changed: hence, making these changes in the expression here exhibited, we shall have for n, in terms of l, d, and s, the value

$$n=\frac{1}{2d}\{d+2l\pm\sqrt{[(2l+d)^2-8ds]}\}.$$

33. Given either of the extreme terms (a or l) of an arithmetical series, together with the sum and difference, to find the other extreme term.

Ans. $\begin{cases} l=-\frac{1}{2}d\pm\sqrt{\{(a-\frac{1}{2}d)^2+2ds\}} \\ a=\ \ \frac{1}{2}d\pm\sqrt{\{(l+\frac{1}{2}d)^2-2ds\}} \end{cases}$

either of which expressions may be deduced from the other upon the principle adverted to above.

(109.) Every equation, containing only two unknown terms, may be reduced to a quadratic, provided the index of the unknown quantity in one term be double its index in the other; for, by putting y for the lowest power or root of the unknown, y^2 will be the highest; so that the equation will become a quadratic.

EXAMPLES.

1. Given $x^n - 2ax^{\frac{n}{2}} = b$, to find the values of x.

Completing the square,* $x^n - 2ax^{\frac{n}{2}} + a^2 = a^2 + b$;

$\therefore$ extracting the root, $x^{\frac{n}{2}} - a = \pm \sqrt{a^2 + b}$;

$\therefore x = (a \pm \sqrt{a^2 + b})^{\frac{2}{n}}$

2. Given $x + 5 = \sqrt{x + 5} + 6$, to find the values of x.

Putting $\sqrt{x + 5} = y$, the equation becomes $y^2 = y + 6$;

or, by transposition, $y^2 - y = 6$,

and completing the square, $y^2 - y + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4}$;

$\therefore$ extracting the root, $y - \frac{1}{2} = \pm \sqrt{\frac{25}{4}} = \pm \frac{5}{2}$;

or $y = 3$, or -2;

$\therefore x + 5 \; (= y^2) = 9$, or 4,

and $x = 4$, or -1.

3. Given $\sqrt{x + 21} + \sqrt[4]{x + 21} = 12$, to find the values of x.

Putting $\sqrt[4]{x + 21} = y$, the equation becomes $y^2 + y = 12$;

and completing the square, $y^2 + y + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4}$;

$\therefore$ extracting the root, $y + \frac{1}{2} = \pm \frac{7}{2}$;

$\therefore y = 3$, or -4;

and $x + 21 \; (= y^4) = 81$, or 256;

$\therefore x = 60$, or 235.

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* The actual substitution of y for the unknown term is not necessary, unless it have a compound form, in which case the substitution will often considerably contract the operation, and render it free from that complex appearance which it would otherwise exhibit.

4. Given $2x^2+3x-5\sqrt{2x^2+3x+9}=-3$, to find the values of x.

Adding 9 to each side, $2x^2+3x+9-5\sqrt{2x^2+3x+9}=6$;

and putting $\sqrt{2x^2+3x+9}=y$, the equation becomes

$$y^2-5y=6\,;$$

$\therefore$ completing the square, $y^2-5y+\frac{25}{4}=6+\frac{25}{4}=\frac{49}{4}$;

and extracting the root, $y-\frac{5}{2}=\pm\frac{7}{2}$;

$$\therefore y=6, \text{ or } -1\,;$$

and taking $y=6$, $2x^2+3x+9\ (=y^2)=36$,

$$\text{or } x^2+\tfrac{3}{2}x=\tfrac{27}{2}\,;$$

and completing the square, $x^2+\frac{3}{2}x+\frac{9}{16}=\frac{27}{2}+\frac{9}{16}=\frac{225}{16}$;

$\therefore$ extracting the root, $x+\frac{3}{4}=\pm\frac{15}{4}$;

$$\text{whence } x=3, \text{ or } -\tfrac{9}{2}\,;$$

or, taking $y=-1$, $2x^2+3x+9=1$;

$$\text{or } x^2+\tfrac{3}{2}x=-4\,;$$

and completing the square, $x^2+\frac{3}{2}x+\frac{9}{16}=\frac{-55}{16}$;

$\therefore$ extracting the root, $x+\frac{3}{4}=\pm\frac{\sqrt{-55}}{4}$;

$$\text{whence } x=\frac{-3\pm\sqrt{-55}}{4}.$$

5. Given $(2x+6)^{\frac{1}{2}}+(2x+6)^{\frac{1}{4}}=6$, to find the values of x.

Ans. $x=5$, or $37\frac{1}{2}$.

6. Given $\dfrac{1}{(2x-4)^2}=\frac{1}{8}+\dfrac{2}{(2x-4)^4}$, to find the values of x.

Ans. $x=3$, or 1.

7. Given $3x^{\frac{4}{3}}-\dfrac{5x^{\frac{8}{3}}}{2}+592=0$, to find the values of x.

Ans. $x=8$, or $-(\frac{74}{5})^{\frac{3}{4}}$.

8. Given $(x+12)^{\frac{1}{2}}=6-(x+12)^{\frac{1}{4}}$, to find the values of x.

Ans. $x=4$, or 69.

9. Given $x = \frac{\sqrt{x^4 - a^4}}{a}$, to find the values of x.

Ans. $x = \pm a \frac{\sqrt{2(1 \pm \sqrt{5})}}{2}$.

10. Given $x^{\frac{5}{2}} - x = 56x^{-\frac{1}{2}}$, to find the values of x.

Ans. $x = 4$, or $\sqrt[3]{49}$.

11. Given $3x^2 + x^{\frac{7}{6}} - 3104x^{\frac{1}{3}} = 0$, to find the values of x.

Ans. $x = 64$, or $(-\frac{97}{3})^{\frac{6}{5}}$.

12. Given $[(2x+1)^2 + x]^2 - x = 90 + (2x+1)^2$, to find the values of x.

Ans. $x = \left\{ \begin{matrix} 1 \\ \text{or } -2\frac{1}{4} \end{matrix} \right\}$, or $x = -\frac{5}{8} \pm \frac{3\sqrt{-15}}{8}$.

ANOTHER METHOD OF SOLVING QUADRATICS.

(110.) Let the equation $ax^2 \pm bx = c$ be multiplied by $4a$, then $4a^2x^2 \pm 4abx = 4ac$; and if b^2 be added to each side, the equation becomes

$$4a^2x^2 \pm 4abx + b^2 = 4ac + b^2.$$

$$\text{or, } (2ax)^2 \pm 2b\,(2ax) + b^2 = 4ac + b^2.$$

Now the first side is evidently a square, viz. $(2ax \pm b)^2$, whence

$$2ax \pm b = \pm \sqrt{4ac + b^2}, \therefore x = \frac{\pm \sqrt{4ac + b^2} \mp b}{2a}.$$

Hence the following rule:

(111.) 1. Having transposed the unknown terms to one side of the equation, and the known terms to the other, multiply each side by 4 times the coefficient of the unknown square.

2. Add the square of the coefficient of the simple power of the unknown, in the proposed equation, to both sides, and the unknown side will then be a complete square.

3. Extract the root, and the value of the unknown quantity is obtained as before.*

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* This method is taken from the Bija Ganita, a Hindoo treatise on Algebra, translated from a Persian copy by Mr. Strachey. For an account of this curious work, see Dr. Hutton's Tracts, vol. ii, p. 162.

EXAMPLES.

1. Given $3x^2 + 5x - 8 = 34$, to find the values of x.

By transposition, $3x^2 + 5x = 42$;

and multiplying by 4×3, or 12, $36x^2 + 60x = 504$;

and completing the square, by adding 5^2,

$$36x^2 + 60x + 25 = 529;$$

$\therefore$ extracting the root, $6x + 5 = \pm 23$;

$$\text{whence } x = \frac{\pm 23 - 5}{6} = 3, \text{ or } -4\tfrac{2}{3}.$$

2. Given $x^2 + 6x + 4 = 22 - x$, to find the values of x.

By transposition, $x^2 + 7x = 18$;

and multiplying by 4, $4x^2 + 28x = 72$;

$\therefore$ completing the square, $4x^2 + 28x + 49 = 121$,

and extracting the root, $2x + 7 = \pm 11$;

$$\text{whence } x = \frac{\pm 11 - 7}{2} = 2, \text{ or } -9.$$

3. Given $8x^2 - 7x + 6 = 171$, to find the values of x.

By transposition, $8x^2 - 7x = 165$;

and multiplying by 4×8, or 32, $256x^2 - 224x = 5280$;

$\therefore$ completing the square, $256x^2 - 224x + 49 = 5329$;

and extracting the root, $16x - 7 = \pm 73$;

$$\therefore x = \frac{\pm 73 + 7}{16} = 5, \text{ or } -\tfrac{33}{8}.$$

4. Given $\sqrt{x + 12} = \dfrac{12}{\sqrt{x + 5}}$, to find the values of x.

Squaring each side, $x + 12 = \dfrac{144}{x + 5}$;

and multiplying by $x + 5$, $x^2 + 17x + 60 = 144$;

$\therefore$ by transposition, $x^2 + 17x = 84$;

and multiplying by 4, $4x^2 + 68x = 336$;

$\therefore$ completing the square, $4x^2 + 68x + 289 = 625$;

and extracting the root, $2x + 17 = \pm 25$;

$$\therefore x = \frac{\pm 25 - 17}{2} = 4, \text{ or } -21.$$

(112.) It will have been perceived, from the preceding solutions, that in equations of the form $ax^2 \pm b = c$, whenever b and c are not both divisible by a, or, being divisible, whenever the quotient of b by a is *odd*, this second method is more commodious than the former, since by that fractions would unavoidably enter the operation in the cases here supposed; but it is the peculiar advantage of the present method that it always precludes the introduction of fractions, however the coefficients of the proposed equation be related; so that when fractions are once removed, by the preparatory operations, they are effectually excluded throughout the solution. It will be unnecessary to add any more examples illustrative of this method, as those already given (Art. 108) will also suffice for this purpose.

We may, however, here point out an obvious simplification in the process, which it will be worth while to attend to in practice. It appears, from the foregoing general formula, that any quadratic $ax^2 \pm bx = c$ is reducible to the simple equation

$$2ax \pm b = \pm \sqrt{4ac + b^2},$$

which reduced form may in practice be written down at once from the proposed equation, without the aid of any intermediate steps: for, if we double the first coefficient in the proposed equation, we shall have the proper coefficient for x in the reduced equation; and if to the first term thus found we connect, with its proper sign, the second coefficient in the proposed, the first side of the reduced equation will be formed.* The second side will be had by multiplying the absolute term (that is, the second side,) in the proposed, by four times the first coefficient, adding to the result the square of the second coefficient, and covering the whole with the sign of the square root. As an illustration, take Example 1, Art. 111, which, after transposition, is

$$3x^2 + 5x = 42;$$

* The more advanced student will at once see that this first side is the side of the *limiting equation* to the proposed.

then, forming each side of the reduced equation as above directed, we get immediately

$$6x + 5 = \pm \sqrt{12 \times 42 + 25},$$

$$\text{that is, } 6x + 5 = \pm 23;$$

$$\therefore x = 3, \text{ or } -4\tfrac{2}{3}.$$

The second example, after transposition, is

$$x^2 + 7x = 18,$$

$$\therefore 2x + 7 = \pm \sqrt{4 \times 18 + 49} = \pm 11;$$

$$\therefore x = 2, \text{ or } -9.$$

And in like manner may all the other examples be solved with the same facility: the student is recommended to apply the method to some of these, by way of practice.

When the coefficients and absolute term in a quadratic equation are very large numbers, the solution may be more expeditiously obtained by the method explained in *The Analysis and Solution of Cubic and Biquadratic Equations*, which forms a supplement to the present volume.*

(113.) But, without proceeding to the actual solution of the equation, we can always discover the character of the roots which that solution would furnish, by a simple inspection of the second member of the reduced form

$$2ax \pm b = \pm \sqrt{4ac + b^2},$$

or rather of the coefficients a, b, c, which enter into it. For if these be such that $4ac + b^2$ be *negative*, which can happen, however, only when a and c have opposite signs, the two roots will evidently be imaginary; but if this condition have not place, then we may pronounce the roots to be real. Hence, in order that

* The student may, if he please, pass now to the "Questions" at article (116); but the intermediate articles, which are devoted to a general discussion of the constitution of a quadratic equation, deserve perusal, as well on account of the intrinsic interest of the subject, as because of its connexion with more advanced analytical inquiries.

the roots may be real, the coefficients of the equation must satisfy one or other of the conditions $4ac + b^2 = 0$, or $4ac + b^2 =$ *a positive number*. If the former have place, then, from the general expression at page 139, the roots must not only be real but equal to one another; if the latter have place, they will be real and unequal.

An appeal to the foregoing tests will thus enable us to discover the character of the roots from examining the original coefficients; and will therefore save us the trouble of actual solution, whenever those roots happen to be imaginary.

The *criteria* just exhibited are usually framed in reference to the general quadratic equation, when all its terms, both known and unknown, are arranged on one side of the equation, leaving zero only on the other side; thus

$$ax^2 + bx + c = 0 \quad \ldots\ldots \quad [A],$$

where a, b, c are considered to be any positive or negative numbers whatever, the *plus* signs serving merely to connect the terms together. Conformably to this arrangement, the following are the criteria for determining the character of the roots.*

When $b^2 - 4ac$ is *negative*, the roots are *imaginary*.
$b^2 - 4ac$ is *positive* .... *real and unequal*.
$b^2 - 4ac = 0$ .... *real and equal*.

As observed above, the letters a, b, c represent the coefficients, *signs and all*: and it may be well here to remark, that this comprehensive signification is always to be given to our symbols when the things they represent are unknown or arbitrary, as well in sign as in numerical value.

It is on this account that, after extracting the root in the solution of a quadratic equation, we abstain from prefixing any sign to the unknown side, and direct our attention solely to the known side for the *complete interpretation* of the former.

* When the third criterion is satisfied, the two equal roots are each $-\frac{b}{2a}$, as is plain from the general formulæ in next page.

(114.) The preceding inferences respecting the nature of the roots of a quadratic equation have been deduced solely from attending to that part of the general expression for those roots which is covered by the radical sign. If we submit the entire expression to examination, we shall obtain a still further insight into the character of the roots; we shall not only be enabled to determine when they are real and when they are imaginary, but further—in the case of real roots—whether they are both positive, both negative, or one positive and the other negative.

The general expression adverted to furnishes, for the two roots of [A], the values

$$x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

By *adding* these values together, we find that

$$\textit{the sum of the roots} = -\frac{b}{a},$$

and by *multiplying* them together—recollecting that the sum multiplied by the difference of two quantities gives the difference of their squares—we learn that

$$\textit{the prod. of the roots} = \frac{c}{a},$$

These conclusions enable us at once to infer the algebraic signs of the roots when the criterion has shown them to be real. For regarding the coefficient a of any quadratic equation [A] to be positive, since it can always be rendered positive, we infer from the above that

When c is *negative*, the roots are *real* and of *opposite signs.*
When c is *positive* and b *negative*, the roots (if real) are both *positive.*
When c and b are both *positive*, the roots (if real) are both *negative.*

The following examples will serve as practical illustrations of these inferences, the character of the roots being determined simply from inspecting the coefficients.

Equations.	*Character of the roots.*
$2x^2 + 3x - 7 = 0$	one + and one −
$5x^2 - 12x + 6 = 0$	both +
$3x^2 + 8x + 5 = 0$	both −
$3x^2 \pm 8x + 6 = 0$	imaginary.
$4x^2 + 12x + 9 = 0$	both −, and *equal.*

(115) Since, as just proved, the sum of the roots of the equation [A], or which is the same thing, of the equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \ldots\ldots [B]$$

is the coefficient of x here exhibited, with its sign changed, and their product equal to the absolute term $\frac{c}{a}$, it follows that if we represent these roots by r, r' the product $(x - r)(x - r')$ must be identical to the first member of [B], seeing that this product is $x^2 - (r + r')x + rr'$, the coefficient of x, with its sign changed, being the *sum* of the roots, and the absolute term their *product*, as in [B]; so that the significant side of every quadratic equation, when reduced to the form [B], is compounded of two single factors, which, when separately equated to zero, furnish the roots of that quadratic: when the roots are found or given, the component factors are x *minus* one root, and x *minus* the other. These factors multiplied by a furnish, of course, the left hand member of [A]. We thus learn that every quadratic expression may be decomposed into its simple factors by equating that expression to zero, and then finding the roots r, r', of the equation thus formed.

(116.) QUESTIONS PRODUCING QUADRATIC EQUATIONS INVOLVING BUT ONE UNKNOWN QUANTITY.

QUESTION I.

It is required to find two numbers, whose difference shall be 12, and product 64.

Let x be the less number;

then $x + 12$ is the greater:

also by the question, $x(x+12)=64$,

that is, $x^2+12x=64$;

$\therefore$ completing the square, $x^2+12x+36=100$;

and extracting the root, $x+6=\pm 10$;

$\therefore x=\pm 10-6=4$, or -16:

hence the numbers are either 4 and 16, or -16 and -4.

QUESTION II.

Having sold a commodity for 56*l.*, I gained as much per cent. as the whole cost me. How much, then, did it cost?

Suppose it cost x pounds;

then the gain was $56-x$;

and by the question, $100 : x :: x : 56-x$;

$\therefore x^2=5600-100x$;

or, by transposition, $x^2+100x=5600$;

and completing the square, $x^2+100x+2500=8100$:

$\therefore$ extracting the root, $x+50=\pm 90$;

whence $x=40$, or -140:

$\therefore$ the commodity cost 40*l.*: the other value of x is inadmissible.

QUESTION III.

A company at a tavern had 8*l.* 15*s.* to pay; but before the bill was paid, two of them went away, when those who remained had, in consequence, 10*s.* each more to pay. How many persons were in company at first?

Let x be the number;

then, $\frac{175}{x}$ is the number of shillings each had to pay at first;

and by the question, $\frac{175}{x}+10$ is the number each had to pay after two had gone;

$$\therefore \left(\frac{175}{x} + 10\right)(x - 2) = 175;$$

that is, $\frac{175x - 350}{x} + 10x - 20 = 175;$

$$\therefore 175x - 350 + 10x^2 = 195x;$$

$$\text{or } 10x^2 - 20x = 350;$$

$$\therefore x^2 - 2x = 35;$$

and completing the square, $x^2 - 2x + 1 = 36;$

$\therefore$ extracting the root, $x - 1 = \pm 6;$

whence $x = 7$, or $-5;$

$\therefore$ there were seven persons at first.*

QUESTION IV.

A person travels from a certain place at the rate of one mile the first day, two the second, three the third, and so on; and, in six days after, another sets out from the same place, in order to overtake him, and travels uniformly at the rate of fifteen miles a day. In how many days will they be together?

Let x be the number of days;

Then the first will have travelled $x + 6$ days;

and (Art. 72, Theo. 5, chap. III.), $(x + 7)\frac{x+6}{2}$ is the distance gone: also $15x$ is the distance the second travels;

$$\therefore (x + 7)\frac{x + 6}{2} = 15x;$$

$$\text{and } \therefore x^2 + 13x + 42 = 30x;$$

or, by transposition, $x^2 - 17x = -42;$

and completing the square (Art. 111), $4x^2 - 68x + 289 = 121;$

$\therefore$ extracting the root, $2x - 17 = \pm 11;$

$$\text{whence } x = \frac{\pm 11 + 17}{2} = 14, \text{ or } 3;$$

hence it appears that they will be together 3 days after the second sets out, who will then overtake the first, and be overtaken by him again in 11 days after, or 14 from the time of the second setting out.

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* For a simpler solution, see *Catechism of Algebra*, Part II.

QUESTION V.

A vintner sold 7 dozen of sherry and 12 dozen of claret for 50*l.*, and finds that he has sold 3 dozen more of sherry for 10*l.* than he has of claret for 6*l.* Required the price of each.

Let x be the price of a dozen of sherry in pounds;

$$\text{then } \frac{10}{x} = \text{the no. of doz. of sherry for } 10l.$$

and by the question, $\frac{10}{x} - 3 = \frac{10-3x}{x} =$ dozens of claret for 6*l.*

$\therefore\ 6 \div \frac{10-3x}{x} = \frac{6x}{10-3x} =$ the price of a dozen of claret;

$$\text{whence } 7x + \frac{72x}{10-3x} = 50;$$

$$\text{or } 70x - 21x^2 + 72x = 500 - 150x;$$

$\therefore$ by transposition, $292x - 21x^2 = 500$;

$$\text{or } x^2 - \tfrac{292}{21}x = -\tfrac{500}{21};$$

and completing the square, $x^2 - \frac{292}{21}x + (\frac{146}{21})^2 = \frac{10816}{(21)^2}$;

$\therefore$ extracting the root, $x - \frac{146}{21} = \pm \frac{104}{21}$;

$$\text{whence } x = 2, \text{ or } \tfrac{250}{21};$$

$\therefore$ the price of a dozen of sherry was 2*l.* and of a dozen of claret $\left(= \frac{6x}{10-3x} =\right)$ 3*l.* If the other value of x be taken, then the number expressing the dozens of claret will be negative: it is therefore inadmissible.

QUESTION VI.

It is required to find two numbers, such that their sum, product, and difference of their squares shall be all equal.

Let the numbers be represented by x and $x+1$, then one condition will necessarily be fulfilled, for the sum, $x + (x+1)$, and the difference of the squares, $(x+1)^2 - x^2$, are each $2x+1$. We have

therefore only to satisfy the remaining condition, that is, to solve the equation

$$x^2 + x = 2x + 1,$$

or by transposition,

$$x^2 - x = 1;$$

$$\text{hence (p. 141) } 2x - 1 = \pm\sqrt{4+1} = \pm\sqrt{5};$$

$$\left.\begin{aligned} \therefore x &= \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5} \\ \text{and } x + 1 &= \tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{5} \end{aligned}\right\} \text{the numbers required.}$$

7. It is required to find two numbers, whose sum shall be 14, such, that 18 times the greater shall be equal to 4 times the square of the less. Ans. 6 and 8.

8. Divide the number 48 into two such parts that their product may be 432. Ans. 36 and 12.

9. Divide the number 24 into two such parts that their product may be equal to 35 times their difference. Ans. 10 and 14.

10. What number is that which exceeds its square root by $48\frac{3}{4}$. Ans. $56\frac{1}{4}$.

11. It is required to find two numbers, the first of which may be to the second as the second is to 16; and the sum of their squares equal to 225. Ans. 9 and 12.

12. A person bought some sheep for 72*l.*, and found that if he had bought 6 more for the same money, he would have paid 1*l.* less for each. How many did he buy, and what was the price of each?

Ans. The number of sheep was 18, and the price of each 4*l.*

13. A merchant sold a quantity of brandy for 39*l.*, and gained as much per cent. as it cost him. What was the price of the brandy?

Ans. 30*l.*

14. In a parcel containing 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins; and each copper coin is worth as many pence as there are silver coins; and the whole is worth 18*s*. How many are there of each?

Ans. 6 silver coins, and 18 copper coins; or 18 silver and 6 copper.

15. A traveller sets out for a certain place, and travels one mile the first day, two the second, three the third, and so on: in 5 days afterwards another sets out, and travels 12 miles a day. How long and how far must he travel to overtake the first?

Ans. He must travel 3 days, or 36 miles.

16. What two numbers are those whose sum, multiplied by their product, is equal to 12 times the difference of their squares; and which are to each other in the ratio of 2 to 3?

Ans. 4 and 6.

17. A person being asked his age, said, "The number representing my age is equal to 10 times the sum of its two digits; and the square of the left hand digit is equal to $\frac{1}{5}$ of my age." Required the person's age?

Ans. 20.

18. Two partners, *A* and *B*, gained 18*l.* by trade. *A*'s money was in trade 12 months, and he received for his principal and gain 26*l.*: also *B*'s money, which was 30*l.*, was in trade 16 months. What money did *A* commence with?

Ans. 20*l.*

19. The joint stock of two partners, *A* and *B*, was 416*l.* *A*'s money was in trade 9 months, and *B*'s 6 months: when they shared stock and gain, *A* received 228*l.*, and *B* 252*l.* What was each man's stock?

Ans. *A*'s stock was 192*l.*, and *B*'s 224*l.*

20. Required the dimensions of a rectangular field, whose length may exceed its breadth by 16 yards, and whose surface may measure 960 square yards.

Ans. Length 40 yards, breadth 24 yards.

21. The plate of a looking-glass is 18 inches by 12, and it is to be surrounded by a plain frame of uniform width, and of surface equal to that of the glass. Required the width of the frame.

Ans. 3 inches.

22. The difference between the hypotenuse and base of a right-angled triangle is 6, and the difference between the hypotenuse and perpendicular is 3. What are the sides?

Ans. 15, 9, and 12.

23. There are three numbers in geometrical proportion; the sum of the first and second is 15, and the difference of the second and third is 36. What are the numbers?

Ans. 3, 12, and 48.

24. It is found, by experiment, that bodies in falling to the earth pass through about $16\frac{1}{12}$ feet in the first second of their motion, and it is known that the spaces passed through from the commencement of motion are as the squares of the intervals elapsed. Suppose, then, that a drop of rain be observed to fall through 595 feet during the last second of its descent, required the height from which it fell; the resistance of the air being neglected?

Ans. $5804\frac{1}{3}$ feet.

ON QUADRATICS INVOLVING TWO UNKNOWN QUANTITIES.

(117.) Equations containing two unknown quantities, in the form of quadratics, cannot be solved, *generally*, by any of the preceding rules, as their solution, in many instances, can only be obtained by means of equations of higher degrees: in several cases, however, the solution may be effected by help of the foregoing methods. These cases we shall now explain.

(118.) *When one of the given Equations is in the form of a Simple Equation.*

Find the value of one of the unknown quantities in the simple equation, in terms of the other and known quantities, and substitute this value for that quantity in the other equation, which will then be a quadratic containing only one unknown quantity.

EXAMPLES.

1. Given $\left\{\begin{array}{l} 2x + y = 10 \\ 2x^2 - xy + 3y^2 = 54 \end{array}\right\}$, to find the values of x and y.

From the first equation,

$$x = \frac{10 - y}{2}, \text{ whence } 2x^2 = \frac{100 - 20y + y^2}{2},$$

$$\text{and } xy = \frac{10y - y^2}{2};$$

$\therefore$ the second equation becomes, by substitution,

$$\frac{100 - 20y + y^2}{2} - \frac{10y - y^2}{2} + 3y^2 = 54;$$

and clearing this equation of fractions,

$$100 - 20y + y^2 - 10y + y^2 + 6y^2 = 108;$$

and by transposition,

$$8y^2 - 30y = 8, \text{ or } y^2 - \tfrac{15}{4}y = 1;$$

$\therefore$ completing the square,

$$y^2 - \tfrac{15}{4}y + \tfrac{225}{64} = \tfrac{289}{64};$$

and extracting the root,

$$y - \tfrac{15}{8} = \pm \tfrac{17}{8};$$

$$\therefore y = 4, \text{ or } -\tfrac{1}{4},$$

$$\text{and } x = 3, \text{ or } \tfrac{41}{8}.$$

2. Given $\left\{ \begin{array}{r} \dfrac{4x + 2y}{3} = 6 \\ xy = 10 \end{array} \right\}$, to find the values of x and y.

From the first equation

$$x = \frac{9 - y}{2},$$

$$\therefore xy = \frac{9y - y^2}{2} = 10.$$

hence

$$9y - y^2 = 20, \text{ or } y^2 - 9y = -20;$$

and completing the square,

$$4y^2 - 36y + 81 = 1;$$

$\therefore$ extracting the root,

$$2y - 9 = \pm 1;$$

$$\text{whence } y = \frac{\pm 1 + 9}{2} = 5, \text{ or } 4;$$

$$\text{and } x = \frac{9 - y}{2} = 2, \text{ or } 2\tfrac{1}{2}$$

Another Solution.

From the first equation, $2x + y = 9$.
Squaring this, we have

$$4x^2 + 4xy + y^2 = 81.$$

Multiplying the second by 8, $\quad 8xy = 80;$

subtracting $\quad 4x^2 - 4xy + y^2 = 1;$

extracting the root, $\quad 2x - y = \pm 1;$

from the first, $\quad 2x + y = 9,$

adding $$4x = 10 \text{ or } 8\,;$$

subtracting, $$2y = 8 \text{ or } 10\,,$$

hence $$x = 2\tfrac{1}{2} \text{ or } 2\,;\ y = 4 \text{ or } 5.$$

3. Given $\left\{ \begin{array}{l} \dfrac{10x+y}{xy} = 3 \\ y - x = 2 \end{array} \right\}$, to find the values of x and y.

From the second equation,
$$y = x + 2,$$

and from the first,
$$10x + y = 3xy.$$

Substituting in this the value of y, just found,
$$10x + x + 2 = 3x^2 + 6x\,;$$

$\therefore$ by transposition,
$$3x^2 - 5x = 2:$$

hence (p. 141)
$$6x - 5 = \pm\sqrt{24 + 25}\,;$$
$$\therefore\ x = \frac{5 \pm 7}{6} = 2 \text{ or } -\tfrac{1}{3}.$$

Consequently, $y = x + 2 = 4$ or $1\tfrac{2}{3}$.

4. Given $\left\{ \begin{array}{l} 4xy = 96 - x^2y^2 \\ x + y = \ 6 \end{array} \right\}$, to find the values of x and y.

From the first equation, by transposition,
$$x^2y^2 + 4xy = 96\,;$$

and substituting for x its value $6 - y$, as obtained from the second, we have
$$(6 - y)^2y^2 + 4\,(6 - y)y = 96\,;$$

or putting $(6 - y)\,y = z$,
$$z^2 + 4z = 96\,;$$

$\therefore$ completing the square,
$$z^2 + 4z + 4 = 100\,;$$

and extracting the root,
$$z + 2 = \pm 10\,;$$

$$\therefore x;\text{ or } 6y - y^2, = 8, \text{ or } -12;$$

$$\therefore y^2 - 6y = -8, \text{ or } 12;$$

and completing the square,

$$y^2 - 6y + 9 = 1, \text{ or } 21;$$

$\therefore$ extracting the root,

$$y - 3 = \pm 1, \text{ or } \pm\sqrt{21};$$

whence $y = 4$, or 2; or $3 \pm \sqrt{21}$; and $x\ (= 6 - y) = 2$, or 4;

or $3 \mp \sqrt{21}$.

5. Given $\left\{\begin{array}{l} \dfrac{x}{y^2} = 2 \\ \frac{1}{3}(x - y) = 5 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 18 \text{ or } 12\frac{1}{2}, \\ y = 3 \text{ or } -2\frac{1}{2}. \end{array}\right.$

6. Given $\left\{\begin{array}{r} x + 4y = 14 \\ y^2 - 2y + 4x = 11 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 2 \text{ or } -46, \\ y = 3 \text{ or } \quad 15. \end{array}\right.$

7. Given $\left\{\begin{array}{l} 2x + y = 22 \\ \dfrac{xy}{2} + y^2 = 16 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 8, \text{ or } \quad 17\frac{3}{5}, \\ y = 6, \text{ or } -13\frac{1}{5}. \end{array}\right.$

8. Given $\left\{\begin{array}{l} x = 15 + y \\ y^3 = \dfrac{xy}{2} \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 18, \text{ or } 12\frac{1}{2}, \\ y = 3, \text{ or } -2\frac{1}{2}. \end{array}\right.$

9. Given $\left\{\begin{array}{l} x + 3y = 16 \\ 3x^2 + 2xy - y^2 = -12 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 1, \text{ or } -7\frac{5}{8}, \\ y = 5, \text{ or } \quad 7\frac{7}{8}. \end{array}\right.$

10. Given $\left\{\begin{array}{l} x + y : x - y :: 13 : 5 \\ x + y^2 = 25 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 9, \text{ or } -14\frac{1}{16}, \\ y = 4, \text{ or } -\ 6\frac{1}{4}. \end{array}\right.$

11. Given $\left\{\begin{array}{c} \dfrac{x^2}{y^2}+\dfrac{4x}{y}=\dfrac{85}{9} \\ x-y=2 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=5, \text{ or } \quad 1\frac{7}{10}, \\ y=3, \text{ or } -\frac{3}{10}. \end{array}\right.$

(119.) If one of the equations consist of the sum or product of the unknown quantities, and the other equation of either the sum of their squares, the sum of their cubes, the sum of their fourth powers, or the sum of their fifth powers, then the solution is obtained by employing a mode somewhat different from that above given, as in the following general examples:

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y=a \\ x^2+y^2=b \end{array}\right\}$, to find the values of x and y.

By squaring the first equation,

$$x^2+2xy+y^2=a^2;$$

and subtracting the second, $x^2 \qquad + y^2 = b,$

we have . . . . $2xy \qquad = a^2-b$:

Also, subtracting this from the second equation,

$$x^2-2xy+y^2=2b-a^2;$$

and, since the first side of this equation is $(x-y)^2$, we have, by extracting the root,

$$x-y=\pm\sqrt{2b-a^2};$$

but $x+y=a$; therefore

$$(x+y)+(x-y)=2x=a\pm\sqrt{2b-a^2},$$

$$\text{or } x=\frac{a\pm\sqrt{2b-a^2}}{2}$$

and $(x+y)-(x-y)=2y=a\mp\sqrt{2b-a^2}$,

$$\text{or } y=\frac{a\mp\sqrt{2b-a^2}}{2}.$$

Or thus :

Put $x = s + z$, and $y = s - z$; then $x + y = 2s$, or $s = \dfrac{a}{2}$;

$$\therefore x^2 = s^2 + 2sz + z^2,$$

$$\text{and } y^2 = s^2 - 2sz + z^2;$$

$$\therefore \text{ by addition, } x^2 + y = 2s^2 + 2z^2 = b:$$

$$\text{whence } z^2 = \frac{b - 2s^2}{2}, \text{ and } \therefore z = \pm\sqrt{\frac{b-2s^2}{2}}$$

$$\text{and } x = s + z = s \pm \sqrt{\frac{b-2s^2}{2}};$$

$$\text{also } y = s - z = s \mp \sqrt{\frac{b-2s^2}{2}};$$

$\therefore$ by restoring the value of s,

$$x = \frac{a}{2} \pm \sqrt{\frac{b - \frac{a^2}{2}}{2}} = \frac{a \pm \sqrt{2b - a^2}}{2},$$

$$\text{and } y = \frac{a}{2} \mp \sqrt{\frac{b - \frac{a^2}{2}}{2}} = \frac{a \mp \sqrt{2b - a^2}}{2}, \text{ as before.}$$

2. Given $\left\{ \begin{array}{l} x + y = a \\ x^3 + y^3 = c \end{array} \right\}$, to find the values of x and y.

By cubing the first equation,

$$x^3 + 3x^2y + 3xy^2 + y^3 = a^3;$$

subtracting the second, $\quad x^3 + y^3 \quad = c,$

$$\text{we have} \ldots\ldots \quad 3x^2y + 3xy^2 = a^3 - c,$$

or $3(x+y)xy = 3axy = a^3 - c$; $\therefore x = \dfrac{a^3 - c}{3ay}$;

and, by substitution,

$$\frac{a^3 - c}{3ay} + y = a, \text{ or } a^3 - c + 3ay^2 = 3a^2y;$$

$\therefore$ by transposing, and dividing by $3a$,

$$y^2 - ay = \frac{c - a^3}{3a};$$

and completing the square,

$$y^2 - ay + \frac{a^2}{4} = \frac{c - a^3}{3a} + \frac{a^2}{4} = \frac{4c - a^3}{12a};$$

$\therefore$ extracting the root,

$$y - \frac{a}{2} = \pm\sqrt{\frac{4c - a^3}{12a}}, \text{ and } y = \frac{a}{2} \pm \sqrt{\frac{4c - a^3}{12a}};$$

$$\therefore x = a - y = \frac{a}{2} \mp \sqrt{\frac{4c - a^3}{12a}}.$$

Or thus:

Putting $x = s + z$, and $y = s - z$, as in the preceding example, we have

$$x^3 = s^3 + 3s^2z + 3sz^2 + z^3$$
$$y^3 = s^3 - 3s^2z + 3sz^2 - z^3;$$

$$\therefore \text{ by addition, } x^3 + y^3 = 2s^3 + 6sz^2 = c;$$

$$\text{whence } z^2 = \frac{c - 2s^3}{6s}, \text{ and } z = \pm\sqrt{\frac{c - 2s^3}{6s}};$$

$$\therefore x = s \pm \sqrt{\frac{c - 2s^3}{6s}}; \text{ and } y = s \mp \sqrt{\frac{c - 2s^3}{6s}};$$

$\therefore$ by restoring the value of s,

$$x = \frac{a}{2} \pm \sqrt{\frac{c - \frac{a^3}{4}}{3a}} = \frac{a}{2} \pm \sqrt{\frac{4c - a^3}{12a}};$$

$$\text{and } \therefore y = \frac{a}{2} \mp \sqrt{\frac{4c - a^3}{12a}} \text{ as before.}$$

3. Given $\left\{ \begin{array}{l} x + y = a \\ x^4 + y^4 = d \end{array} \right\}$, to find the values of x and y.

By involving the first equation to the fourth power,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = a^4$$

and subtracting the second, $x^4 + y^4 = d$

there results . . . $4x^3y + 6x^2y^2 + 4xy^3 = a^4 - d;$

$\therefore$ dividing by xy . . $4x^2 + 6xy + 4y^2 = \frac{a^4 - d}{xy}$:

Now $4(x+y)^2 = 4x^2 + 8xy + 4y^2 = 4a^2$

hence, by subtraction, $2xy = 4a^2 - \frac{a^4 - d}{xy}$,

$$\text{or } 2x^2y^2 = 4a^2xy - a^4 + d;$$

$\therefore$ by transposition and division,

$$x^2y^2 - 2a^2xy = \frac{d - a^4}{2};$$

and completing the square,

$$x^2y^2 - 2a^2xy + a^4 = \frac{d - a^4}{2} + a^4 = \frac{d + a^4}{2};$$

$\therefore$ extracting the root,

$$xy - a^2 = \pm\sqrt{\frac{d + a^4}{2}}, \text{ and } xy = a^2 \pm \sqrt{\frac{d + a^4}{2}};$$

and putting, for simplicity's sake, $a^2 \pm \sqrt{\frac{d + a^4}{2}} = m$

we have, by substitution,

$$\frac{m}{y} + y = a, \text{ or } m + y^2 = ay, \text{ or } y^2 - ay = -m;$$

$\therefore$ completing the square and extracting the root,

$$y - \frac{a}{2} = \pm\sqrt{\left(\frac{a^2}{4} - m\right)}:$$

$$\text{whence } y = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - m} = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - a^2 \mp \sqrt{\frac{d + a^4}{2}}}$$

$$= \frac{a}{2} \pm \sqrt{-\frac{3a^2}{4} \mp \sqrt{\frac{d + a^4}{2}}};$$

$$\text{and } \therefore x = a - y = \frac{a}{2} \mp \sqrt{-\frac{3a^2}{4} \mp \sqrt{\frac{d + a^4}{4}}}.$$

Or thus:

Putting $x = s + z$, and $y = s - z$, as in the preceding examples, we have

$$x^4 = s^4 + 4s^3z + 6s^2z^2 + 4sz^3 + z^4,$$

$$y^4 = s^4 - 4s^3z + 6s^2z^2 - 4sz^3 + z^4;$$

$\therefore$ by addition, $x^4 + y^4 = 2s^4 + 12s^2z^2 + 2z^4 \qquad = d;$

and dividing by 2, . . $s^4 + 6s^2z^2 + z^4 = \frac{d}{2};$

$$\therefore z^4 + 6s^2z^2 = \frac{d}{2} - s^4;$$

and completing the square, and extracting the root,

$$z^2 + 3s^2 = \pm \sqrt{\left(\frac{d}{2} + 8s^4\right)};$$

$$\therefore z = \pm \sqrt{-3s^2 \pm \sqrt{\left(\frac{d}{2} + 8s^4\right)}};$$

consequently, $x = s \pm \sqrt{-3s^2 \pm \sqrt{\left(\frac{d}{2} + 8s^4\right)}},$

and $y = s \mp \sqrt{-3s^2 \mp \sqrt{\left(\frac{d}{2} + 8s^4\right)}};$

or restoring the value of s,

$$x = \frac{a}{2} \pm \sqrt{-\frac{3a^2}{4} \pm \sqrt{\frac{d + a^4}{2}}},$$

and $y = \frac{a}{2} \mp \sqrt{-\frac{3a^2}{4} \mp \sqrt{\frac{d + a^4}{2}}}$, as before.

4. Given $\left\{\begin{matrix} x+y=a \\ x^5+y^5=e \end{matrix}\right\}$, to find the values of x and y.

By involving the first equation to the fifth power,

$$x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5=a^5;$$

and subtracting the second, $x^5+y^5 \qquad =e;$

we have . . $5x^4y+10x^3y^2+10x^2y^3+5xy^4 \qquad =a^5-e;$

and dividing by $5xy$, $x^3+2x^2y+2xy^2+y^3=\dfrac{a^5-e}{5xy}$;

$$\text{But } (x+y)^3=x^3+3x^2y+3xy^2+y^3=a^3;$$

$\therefore$ by subtraction . . $x^2y+xy^2=a^3-\dfrac{a^5-e}{5xy}$;

$$\text{or } (x+y)\,xy=axy=a^3-\frac{a^5-e}{5xy};$$

$\therefore$ multiplying by $5xy$, and transposing, we have

$$5ax^2y^2-5a^3xy=e-a^5$$

$$\text{or } x^2y^2-a^2xy=\frac{e-a^5}{5a};$$

and, by solving this quadratic, we obtain

$$xy=\frac{a^2}{2}\pm\sqrt{\frac{a^5+4e}{20a}};$$

$\therefore$ calling this value of xy, m, we have, from the equation,

$$x+y=a,$$

$$\frac{m}{y}+y=a, \text{ or } m+y^2=ay, \therefore y^2-ay=-m;$$

$$\therefore y=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-m}=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-\frac{a^2}{2}\pm\sqrt{\frac{a^5+4e}{20a}}}$$

$$=\frac{a}{2}\pm\sqrt{-\frac{a^2}{4}\mp\sqrt{\frac{a^5+4e}{20a}}};$$

$$\text{and } x=a-y=\frac{a}{2}\mp\sqrt{-\frac{a^2}{4}\mp\sqrt{\frac{a^5+4e}{20a}}}$$

Or thus:

Putting $x=s+z$, and $y=s-z$, as in the preceding examples,

$$x^5=s^5+5s^4z+10s^3z^2+10s^2z^3+5sz^4+z^5,$$
$$y^5=s^5-5s^4z+10s^3z^2-10s^2z^3+5sz^4-z^5:$$

$\therefore$ by addition, $x^5+y^5=2s^5+20s^3z^2+10sz^4 \qquad =e,$

$$\text{or } z^4+2s^2z^2+\frac{s^4}{5}=\frac{e}{10s};$$

and by transposing, and completing the square,

$$z^4+2s^2z^2+s^4=\frac{e}{10s}-\frac{s^4}{5}+s^4=\frac{8s^5+e}{10s};$$

$\therefore$ extracting the root, &c.

$$z^2=-s^2\pm\sqrt{\frac{8s^5+e}{10s}}, \therefore z=\pm\sqrt{-s^2\pm\sqrt{\frac{8s^5+e}{10s}}};$$

whence, by restoring the value of s,

$$x=s+z=\frac{a}{2}\pm\sqrt{-\frac{a^2}{4}\pm\sqrt{\frac{\frac{a^5}{4}+e}{5a}}}$$
$$=\frac{a}{2}\pm\sqrt{-\frac{a^2}{4}\pm\sqrt{\frac{a^5+4e}{20a}}};$$
$$\text{and } \therefore y=a-x=\frac{a}{2}\mp\sqrt{-\frac{a^2}{4}\pm\sqrt{\frac{a^5+4e}{20a}}};$$

the same as before.*

5. Given $\left\{\begin{array}{r} x^n+y^n=a \\ xy=b \end{array}\right\}$, to find the values of x and y.

By squaring the first equation,

$$x^{2n}+2x^ny^n+y^{2n}=a^2,$$

4 times the nth power of the second, gives

$$4x^ny^n \qquad =4b^n.$$

---

* If we had given $x+y$, and x^6+y^6, to find x and y, the question would be impossible in quadratics, since, as it is easy to perceive, the operation would lead to a cubic equation; we cannot, therefore, extend the above examples any further.

By subtraction,

$$x^{2n} - 2x^n y^n + y^{2n} \quad = a^2 - 4b^n.$$

Extracting the root,

$$x^n - y^n = \sqrt{a^2 - 4b^n}.$$

Therefore, by adding and subtracting this from the first of the given equations, and then taking the nth root, we have,

$$x = \left\{ \tfrac{1}{2}a + \tfrac{1}{2}\sqrt{a^2 - 4b^n} \right\}^{\frac{1}{n}}$$

$$y = \left\{ \tfrac{1}{2}a - \tfrac{1}{2}\sqrt{a^2 - 4b^n} \right\}^{\frac{1}{n}}$$

6. Given $\left\{ \begin{array}{r} x + y = s \\ xy = p \end{array} \right\}$ to find the values of $x^2 + y^2$, $x^3 + y^3$, $x^4 + y^4$, &c.

By squaring the first equation,

$$x^2 + 2xy + y^2 = s^2;$$

and subtracting twice the second, $\quad 2xy \quad = 2p;$

there results . . . . $\quad x^2 + y^2 = s^2 - 2p.$

$$\begin{aligned} \text{Again, } (x + y)(x^2 + y^2) &= s^3 - 2ps \\ xy(x + y) &= \quad\quad ps \\ \hline \therefore x^3 + y^3 &= s^3 - 3ps \end{aligned}$$

$$\begin{aligned} \text{Also, } (x + y)(x^3 + y^3) &= s^4 - 3ps^2 \\ xy(x^2 + y^2) &= \quad\quad ps^2 - 2p^2 \\ \hline \therefore x^4 + y^4 &= s^4 - 4ps^2 + 2p^2 \end{aligned}$$

In like manner,

$$\begin{aligned} (x + y)(x^4 + y^4) &= s^5 - 4ps^3 + 2p^2s \\ xy(x^3 + y^3) &= \quad\quad ps^3 - 3p^2s \\ \hline \therefore x^5 + y^5 &= s^5 - 5ps^3 + 5p^2s \end{aligned}$$

By continuing this simple process, formulas may be deduced to any extent.* These formulas, it may be remarked, would have enabled us to arrive at simpler solutions to the four preceding questions than those already given. Thus, taking the fourth question, we have by the formula last deduced,

$$a^5 - 5a^3p + 5ap^2 = e$$

$$\therefore 5ap^2 - 5a^3p = e - a^5$$

and, completing the square and extracting the root,

$$p = \tfrac{1}{2}a^2 \pm \tfrac{1}{2}\sqrt{\frac{a^5 + 4e}{5a}} = xy.$$

Now $x - y = \sqrt{a^2 - 4p}$, and half this added to $\frac{1}{2}(x + y)$ or $\frac{1}{2}a$ gives x, and subtracted from it gives y: hence,

$$x = \tfrac{1}{2}a \pm \tfrac{1}{2}\sqrt{\left\{-a^2 \pm 2\sqrt{\frac{a^5 + 4e}{5a}}\right\}}.$$

PARTICULAR EXAMPLES.

1. Given the sum of two numbers equal to 24, and the sum of their squares equal to 306. To find the numbers.

   Ans. 9 and 15.

2. The sum of two numbers is 27, and the sum of their cubes 4941. Required the numbers.

   Ans. 13 and 14.

3. The sum of two numbers is 11, and the sum of their fourth powers 2657. What are the numbers?

   Ans. 4 and 7.

---

* The general expression for the sum of the nth powers would thus be found to be

$$x^n + y^n = s^n - nps^{n-2} + \frac{n(n-3)}{2}p^2s^{n-4} - \frac{n(n-4)(n-5)}{2 \cdot 3}p^3s^{n-6} + \ldots.$$

4. The sum of two numbers is 10, and the sum of their fifth powers 17050. What are the numbers?

Ans. 3 and 7.

5. The sum of two numbers is 47, and their product 546. Required the sum of their squares.

Ans. 1117.

6. The sum of two numbers is 20, and their product 99. Required the sum of their cubes.

Ans. 2060.

7. The sum of two numbers is 19, and their product 78. What is the sum of their fourth powers?

Ans. 29857.

SCHOLIUM.

(120.) It is worthy of remark that the values of x and y, as exhibited in example 5, page 161, lead very readily to a certain expression, known by the name of "*Cardan's Formula*" for the solution of *Cubic Equations*. For if n be equal to 3, the values referred to will be

$$\left.\begin{aligned} x &= \{\tfrac{1}{2}a + \tfrac{1}{2}\sqrt{(a^2 - 4b^3)}\}^{\frac{1}{3}} \\ y &= \{\tfrac{1}{2}a - \tfrac{1}{2}\sqrt{(a^2 - 4b^3)}\}^{\frac{1}{3}} \end{aligned}\right\} \ldots\ldots [A].$$

Now the first of the given equations, which these values of x and y satisfy, is the same as

$$(x + y)^3 - 3xy(x + y) = a;$$

that is, from the second equation,

$$(x + y)^3 - 3b(x + y) = a.$$

Hence the sum of the expressions [A] will represent the roots of this *cubic* equation; or, putting for abridgment z for $x + y$, that sum will express the roots of the equation

$$z^3 - 3bz = a.$$

If instead of b we write $-\tfrac{1}{3}c$, then the roots of the cubic equation

$$z^3 + cz = a \ldots\ldots [B]$$

will be thus expressed, viz.

$$z = \{\tfrac{1}{2}a + \sqrt{(\frac{a^2}{4} + \frac{c^3}{27})}\}^{\frac{1}{3}} + \{\tfrac{1}{2}a - \sqrt{(\frac{a^2}{4} + \frac{c^3}{27})}\}^{\frac{1}{3}}$$

And this is Cardan's formula for the solution of a cubic equation of the form [B], to which form every equation, $x^3 + Px^2 + Qx = N$, may be reduced by substituting in it $z - \frac{P}{3}$ for x.

In the application of this formula the extraction of *two* cube roots may be avoided by writing the second term of it thus:

$$\frac{\frac{1}{3}c}{\{\frac{1}{2}a + \sqrt{(\frac{a^2}{4} + \frac{c^3}{27})}\}^{\frac{1}{3}}}$$

which is allowable, since the second term is produced from this by multiplying numerator and denominator by the *difference* of the terms within the brace. But, however the formula be modified, its application to numerical examples will generally prove tedious, and in some instances (belonging to what is called the *irreducible case*) unsatisfactory. It may, therefore, be advantageously replaced by the short and easy method given in the *Analysis and Solution of Cubic and Biquadratic Equations.*

(121.) *When both Equutions have a Quadratic Form.*

In this case, which includes every possible form, no general method of procedure can be pointed out; and the solution, in most cases, must therefore be left to the ingenuity of the learner. Many equations, however, that come under this case are irresolvable by *quadratics only*, and require equations of the higher degrees, as has been before observed.* When, however, the proposed quadratics are both *homogeneous* as respects the un-

* The most general form under which quadratics containing two unknown quantities can be expressed is this, viz.

$$ax^2 + by^2 + cxy + dx + ey = A,$$
$$a'x^2 + b'y^2 + c'xy + d'x + e'y = A';$$

the solution of which is a branch of the general doctrine of Elimination, a subject too comprehensive to be treated on fully in an elementary

knowns, that is, when every unknown term is of two dimensions, the solution may always be effected by adopting the artifice of substituting for one of the unknowns an unknown multiple of the other; because we shall thus introduce the square of this other into every term, and may therefore eliminate it from the equations. The result of this elimination will be a single quadratic, in which the unknown is the assumed multiplier at first introduced, and the determination of which leads immediately to the solution. The most general form in which a pair of homogeneous quadratics can occur is the following, viz.

$$\left.\begin{aligned} ax^2 + bxy + cy^2 &= d \\ a'x^2 + b'xy + c'y^2 &= d' \end{aligned}\right\}*.$$

To find the values of x and y in these equations, put $x = zy$, and they become

$$\left.\begin{aligned} az^2y^2 + bzy^2 + cy^2 &= d \\ a'z^2y^2 + b'zy^2 + c'y^2 &= d' \end{aligned}\right\}$$

from the first of which we get

$$y^2 = \frac{d}{az^2 + bz + c};$$

and from the second,

$$y^2 = \frac{d'}{a'z^2 + b'z + c'},$$

$$\text{whence } \frac{d}{az^2 + bz + c} = \frac{d'}{a'z^2 + b'z + c'};$$

or, clearing the equations of fractions,

$$a'dz^2 + b'dz + c'd = ad'z^2 + bd'z + cd':$$

a quadratic, from which the values of z may be found, and, consequently, those of y and x may then be determined.

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work like the present. The elimination of equations of the first degree has been already given in Chap. II.; and, for the extension of the theory to equations of the higher degrees, the reader is referred to the treatise on the *Theory of Equations*: the final equation, however, to which the above pair of quadratics leads, is ordinarily of the fourth degree.

* This general form evidently includes a great variety of equations, since it comprehends all those in which any of the coefficients a, b, c, a', b', c', are $=0$; that is, in which any of the terms are absent.

EXAMPLES.

1. Given $\left\{\begin{array}{l} 4x^2-2xy=12 \\ 2y^2+3xy=8 \end{array}\right\}$, to find the values of x and y.

These equations being homogeneous, are resolvable by the above process; therefore, assuming $x=zy$, we have

$$4z^2y^2-2zy^2=(4z^2-2z)y^2=12,$$

$$\text{and } 2y^2+3zy^2=(2+3z)y^2=8;$$

$\therefore$ from the first equation,

$$y^2=\frac{6}{2z^2-z};$$

and from the second,

$$y^2=\frac{8}{2+3z};$$

$$\therefore \frac{6}{2z^2-z}=\frac{8}{2+3z},$$

$$\text{or } \frac{3}{2z^2-z}=\frac{4}{2+3z};$$

$$\text{hence } 6+9z=8z^2-4z;$$

and by transposition,

$$8z^2-13z=6, \text{ or } z^2-\tfrac{13}{8}z=\tfrac{6}{8};$$

$\therefore$ completing the square, and extracting the root,

$$z-\tfrac{13}{16}=\pm\tfrac{19}{16}, \therefore z=2, \text{ or } -\tfrac{3}{8};$$

$$\text{and } y^2=\frac{8}{2+3z}=1, \text{ or } \tfrac{64}{7};$$

$$\therefore y=\pm 1, \text{ or } \pm\sqrt{\tfrac{64}{7}}=\pm\tfrac{8}{7}\sqrt{7}$$

$$\text{and } x=zy=\pm 2, \text{ or } \mp\tfrac{3}{8}.\sqrt{\tfrac{64}{7}}=\mp\tfrac{3}{7}\sqrt{7}.$$

2. Given $\left\{\begin{array}{l} 6x^2+2y^2=5xy+12 \\ 2xy+3x^2=3y^2-3 \end{array}\right\}$, to find the values of x and y.

These equations being homogeneous, substitute, as before, zy for x, and we have

$$6z^2y^2-5zy^2+2y^2 = \{6z^2-5z+2\}y^2 = 12;$$

$$\text{and } 2zy^2-3y^2+3z^2y^2=\{2z-3+3z^2\}y^2=-3;$$

from the first of these equations,

$$y^2 = \frac{12}{6z^2 - 5z + 2};$$

and from the second,

$$y^2 = \frac{-3}{2z - 3 + 3z^2};$$

$$\therefore \frac{12}{6z^2 - 5z + 2} = \frac{-3}{2z - 3 + 3z^2},$$

or $24z - 36 + 36z^2 = -18z^2 + 15z - 6$; $\therefore 54z^2 + 9z = 30$,

$$\therefore 6z^2 + z = \tfrac{10}{3};$$

and completing the square (Art. 111),

$$144z^2 + 24z + 1 = 81;$$

and extracting,

$$12z + 1 = \pm 9;$$

$$\therefore z = \tfrac{2}{3}, \text{ or } -\tfrac{5}{6}:$$

whence $y^2 = 9$, or $\frac{36}{31}$; $\therefore y = \pm 3$, or $\pm \dfrac{6}{\sqrt{31}} = \pm \dfrac{6}{31}\sqrt{31}$;

and $\therefore x = \pm 2$, or $\mp \dfrac{5}{\sqrt{31}} = \mp \dfrac{5}{31}\sqrt{31}$.

3. Given $\left\{ \begin{array}{l} x^2 + xy = 12 \\ xy - 2y^2 = 1 \end{array} \right\}$, to find the values of x and y.

Ans. $\left\{ \begin{array}{l} x = \pm 3, \text{ or } \pm \frac{4}{3}\sqrt{6}, \\ y = \pm 1, \text{ or } \pm \frac{1}{6}\sqrt{6}. \end{array} \right.$

4. Given $\left\{ \begin{array}{l} 3x^2 + xy = 68 \\ 4y^2 + 3xy = 160 \end{array} \right\}$, to find the values of x and y.

Ans. $\left\{ \begin{array}{l} x = \pm 4, \text{ or } \mp \frac{34\sqrt{3}}{9}, \\ y = \pm 5, \text{ or } \pm \frac{16\sqrt{3}}{3}. \end{array} \right.$

5. Given $\left\{ \begin{array}{l} 2x^2 - 3xy + y^2 = 4 \\ 2xy - 3y^2 - x^2 = -9 \end{array} \right\}$, to find the values of x and y.

Ans. $\left\{ \begin{array}{l} x = \pm 3, \text{ or } \mp \frac{1}{6}\sqrt{2}, \\ y = \pm 2, \text{ or } \pm \frac{7}{6}\sqrt{2}. \end{array} \right.$

(122.) MISCELLANEOUS EXAMPLES,

To which the preceding Methods do not immediately apply.

1. Given $\begin{cases} x^2+x+y=18-y^2 \\ xy=6 \end{cases}$, to find the values of x and y.

From the first equation, by transposition,

$$x^2+y^2+x+y=18;$$

and from the second, by multiplication,

$$2xy=12;$$

$$\therefore \text{ by addition, } x^2+2xy+y^2+x+y=30;$$

and substituting in this equation the value of x $(=\frac{6}{y})$, as obtained from the second equation, it becomes

$$(\frac{6}{y}+y)^2+(\frac{6}{y}+y)=30,$$

or putting $\frac{6}{y}+y=z$, it is $z^2+z=30$,

$\therefore$ completing the square,

$$z^2+z+\tfrac{1}{4}=\tfrac{121}{4};$$

and extracting the root,

$$z+\tfrac{1}{2}=\pm\tfrac{11}{2};$$

$$\therefore z, \text{ or } \frac{6}{y}+y, =5, \text{ or } -6;$$

$$\text{whence } 6+y^2=5y, \text{ or } -6y;$$

$$\therefore \begin{cases} y^2-5y=-6, \\ \text{or } y^2+6y=-6; \end{cases}$$

and completing the first square (Art. 111),

$$4y^2-20y+25=1;$$

and extracting,

$$2y-5=\pm 1;$$

$$\therefore y=3, \text{ or } 2;$$

also, completing the second square,

$$y^2+6y+9=3;$$

and extracting,

$$y+3=\pm\sqrt{3};$$

$$\therefore y=-3\pm\sqrt{3};$$

whence $x=\frac{6}{y}=2$, or 3; or $-3\mp\sqrt{3}$.

2. Given $\left\{ \begin{array}{l} x^3y - \ y = 21 \\ x^2y - xy = \ 6 \end{array} \right\}$, to find the values of x and y.

From the first equation,

$$y = \frac{21}{x^3 - 1};$$

and from the second,

$$y = \frac{6}{x^2 - x};$$

$$\therefore \frac{21}{x^3 - 1} = \frac{6}{x^2 - x},$$

$$\text{or dividing by } \frac{3}{x - 1},$$

$$\frac{7}{x^2 + x + 1} = \frac{2}{x}, \therefore 7x = 2x^2 + 2x + 2;$$

and by transposition,

$$2x^2 - 5x = -2;$$

$\therefore$ completing the square (Art. 111),

$$16x^2 - 40x + 25 = 9;$$

and extracting the root,

$$4x - 5 = \pm 3;$$

$$\therefore x = 2, \text{ or } \tfrac{1}{2},$$

$$\text{and } y = \frac{6}{x^2 - x} = 3, \text{ or } -24.$$

3. Given $\left\{ \begin{array}{l} x^2 + 3x + y = 73 - 2xy \\ y^2 + 3y + x = 44 \end{array} \right\}$, to find the values of x and y.

By transposition, the first equation becomes

$$x^2 + 2xy + 3x + y = 73,$$

to which, if the second equation be added, there results

$$x^2 + 2xy + y^2 + 4x + 4y = (x + y)^2 + 4(x + y) = 117;$$

and completing the square,

$$(x + y)^2 + 4(x + y) + 4 = 121;$$

∴ extracting the root,

$$(x+y)+2=\pm 11;$$

$$\therefore x+y=9, \text{ or } -13;$$

$$\text{and } x=9-y, \text{ or } -13-y;$$

and, by substituting these values of x in the second equation, we have

$$y^2+2y+\ \ 9=44,$$

$$\text{or } y^2+2y-13=44;$$

∴ by transposing, and completing the square in the first equation,

$$y^2+2y+1=36;$$

and extracting the root,

$$y+1=\pm 6;$$

$$\therefore y=5, \text{ or } -7;$$

also, by transposing, and completing the square in the second equation,

$$y^2+2y+1=58;$$

and extracting the root,

$$y+1=\pm\sqrt{58};$$

$$\therefore y=-1\pm\sqrt{58};$$

hence the values of y are, $y=5$, or -7; or $-1\pm\sqrt{58}$,

and those of x are ∴ $\quad x=4$, or 16; or $-12\mp\sqrt{58}$.

4. Given $\left\{\begin{array}{r} x^2-y^2-(x+y)=\ 8 \\ (x-y)^2(x+y)=32 \end{array}\right\}$, to find the values of x and y.

Multiplying the first equation by 4, we have

$$4\{x^2-y^2-(x+y)\}=(x-y)^2(x+y);$$

and, dividing this by $x+y$, there results

$$4\,(x-y-1)=(x-y)^2;$$

and by transposition,

$$(x-y)^2-4\,(x-y)=-4;$$

∴ completing the square,

$$(x-y)^2-4\,(x-y)+4=0$$

and extracting the root,

$$(x-y)-2=0;$$

$$\therefore x-y=2;$$

and this value of $x-y$, substituted in the second equation, gives

$$4(x+y)=32, \therefore x+y=8;$$

and by addition, $\left\{\begin{array}{l} x-y=\ 2 \\ x+y=\ 8 \end{array}\right\}$ also, by subtraction, $\left\{\begin{array}{l} x+y=8, \\ x-y=2, \end{array}\right.$

we get $2x=10;$ we get $2y=6;$

whence $x=5$; and $y=3$.

5. Given $\left\{\begin{array}{c} \dfrac{x^2}{y}+\dfrac{y^2}{x}=a \\ x+y=2b \end{array}\right\}$, to find the values of x and y.

Assume $x=z+v$,

and $y=z-v$;

$$\therefore x+y=2z=2b;$$

$$\therefore z=b: \therefore x=b+v,\ y=b-v.$$

Now from the first equation,

$$x^3+y^3=axy \quad \ldots\ldots\ldots\ [1];$$

$$\text{but } x^3=(b+v)^3=b^3+3b^2v+3bv^2+v^3;$$

$$y^3=(b-v)^3=b^3-3b^2v+3bv^2-v^3;$$

$$\therefore x^3+y^3=2b^3+6bv^2 \quad \ldots\ldots\ [2].$$

Again,

$$axy=a(b+v)(b-v)=ab^2-av^2 \quad \ldots\ [3]$$

hence, substituting [2] and [3] in [1], we have

$$2b^3+6bv^2=ab^2-av^2;$$

$$\therefore (a+6b)v^2=ab^2-2b^3;$$

$$\therefore v^2=\frac{b^2(a-2b)}{a+6b};$$

$$\therefore\ v = b\sqrt{\frac{a-2b}{a+6b}};$$

$$\therefore\ x = b + b\sqrt{\frac{a-2b}{a+6b}};$$

$$y = b - b\sqrt{\frac{a-2b}{a+6b}}.$$

The artifice employed in the solution of the above example, which consists in substituting for the unknowns the sum and difference of two other unknowns, will always succeed when, as in the present case, the proposed equations are both *symmetrical*; that is to say, when the unknowns so enter that x and y may be interchanged without essentially altering the equation.

The equations proposed at art. (119) are all symmetrical; and the artifice here adverted to is employed in the solution of most of them.

6. Given $\left\{\begin{array}{l} x^2 + x = \dfrac{12}{y} \\ x^3y + y = 18 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 2, \text{ or } \frac{1}{2}, \\ y = 2, \text{ or } 16. \end{array}\right.$

7. Given $\left\{\begin{array}{l} x^2 + 4y^2 = 256 - 4xy \\ 4y^2 - x^2 = 64 \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = \pm 6, \\ y = \pm 5. \end{array}\right.$

8. Given $x + \dfrac{1}{y} = a$, and $y + \dfrac{1}{x} = b$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = \dfrac{a}{2}\{1 \pm \sqrt{(1 - \dfrac{4}{ab})}\}, \\ y = \dfrac{b}{2}\{1 \pm \sqrt{(1 - \dfrac{4}{ab})}\}. \end{array}\right.$

9. Given $\left\{\begin{array}{l} (x^2+y^2)(x-y) = 51 \\ x^2+y^2) + x = 20 + y \end{array}\right\}$, to find the values of x and y.

Ans. $\left\{\begin{array}{l} x = 4, \text{ or } -1; \text{ or } \dfrac{17 \pm \sqrt{-283}}{2}, \\ y = 1, \text{ or } -4; \text{ or } \dfrac{-17 \pm \sqrt{-283}}{2}. \end{array}\right.$

(123.) QUESTIONS PRODUCING QUADRATIC EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES.

QUESTION I.

It is required to find three numbers, such, that the difference of the first and second shall exceed the difference of the second and third by 6; and that their sum may be 33, and the sum of their squares 467.

Let x be the second number, and y the difference of the second and third; that is, let the third be $x - y$.

then the first, by the question, must be $x + y + 6$;

$\therefore$ their sum $= 3x + 6 = 33$, $\therefore x = 9$;

also $x^2 + (x - y)^2 + (x + y + 6)^2 = 467$;

$$\therefore (x - y)^2 + (x + y + 6)^2 = 386;$$

$$\text{that is, } 2x^2 + 12x + 12y + 2y^2 + 36 = 386,$$

or substituting for x its value $= 9$,

$$306 + 12y + 2y^2 = 386;$$

$$\therefore y^2 + 6y = 40;$$

and completing the square,

$$y^2 + 6y + 9 = 49;$$

$\therefore$ extracting the root,

$$y + 3 = \pm 7,$$

$$\text{and } y = 4, \text{ or } -10;$$

hence the three numbers are 5, 9, and 19; or rather 19, 9, and 5.

QUESTION II.

It is required to find three numbers in geometrical progression, such, that their sum shall be 14, and the sum of their squares 84.

Let $\frac{x}{y}$, x, and xy, be the three numbers;

then, by the question,

$$\frac{x}{y} + x + xy = 14,$$

$$\text{and } \frac{x^2}{y^2} + x^2 + x^2y^2 = 84;$$

∴ from the first equation,

$$\frac{x}{y} + xy = 14 - x;$$

or squaring each side,

$$\frac{x^2}{y^2} + 2x^2 + x^2y^2 = 14^2 - 28x + x^2;$$

$$\therefore \frac{x^2}{y^2} + \ x^2 + x^2y^2 = 14^2 - 28x;$$

and ∴ from the second equation, we have

$$84 = 14^2 - 28x;$$

$$\therefore 6 = 14 - 2x, \text{ and } \therefore x = \frac{14 - 6}{2} = 4;$$

and substituting this value of x in the first equation,

$$\frac{4}{y} + 4 + 4y = 14\ ;$$

$$\therefore 4y^2 - 10y = -4,$$

$$\text{or } y^2 - \tfrac{10}{4}y = -1;$$

and completing the square,

$$y^2 - \tfrac{10}{4}y + \tfrac{25}{16} = \tfrac{9}{16};$$

∴ extracting the root,

$$y - \tfrac{5}{4} = \pm\ \tfrac{3}{4};$$

$$\text{whence } y = 2, \text{ or } \tfrac{1}{2}.$$

∴ the three numbers are 2, 4, and 8.

Another Solution.

Let x and y denote the two extremes, then $\sqrt{xy}$ is the mean, and by the question,

$$x + \sqrt{xy} + y = 14,$$

$$\text{and } x^2 + xy + y^2 = 84.$$

Dividing this equation by the former,

$$x - \sqrt{xy} + y = 6;$$

hence, by addition to the first,

$$x + y = 10;$$

and by subtraction,

$$\sqrt{xy} = 4, \text{ or } xy = 16;$$

consequently,

$$(x + y)^2 - 4xy = 100 - 64 = 36,$$

$$\begin{array}{rl} \therefore x - y & = 6 \\ x + y & = 10 \\ \hline \therefore x = 8, y = & 2 \\ \hline \end{array}$$

hence the numbers are 2, 4, and 8.

QUESTION III.

The sum of four numbers in arithmetical progression is 34, and the sum of their squares 334. What are the numbers?

Let the two means be $x + y$, and $x - y$;

then the extremes will be $x + 3y$, and $x - 3y$;

and their sum $= 4x = 34$, $\therefore x = \frac{17}{2}$;

also the sum of their squares $= 4x^2 + 20y^2 = 334$;

$\therefore$ substituting in this equation the value of x found above, we have

$$289 + 20y^2 = 334;$$

$$\therefore 20y^2 = 45;$$

$$\text{whence } y = \pm\sqrt{\tfrac{9}{4}} = \pm\tfrac{3}{2};$$

$\therefore$ the four numbers are 13, 10, 7, and 4.

QUESTION IV.

The sum of three numbers in harmonical proportion is 13, and the product of their extremes is 18. What are the numbers?

Let the extremes be x and y;

then the mean will be $\frac{2xy}{x+y}$ (art. 80, Ch. III);

and their sum $= x + \frac{2xy}{x+y} + y = 13$;

also the product of the extremes $= xy = 18$;

$\therefore$ by substitution,

$$x + \frac{36}{x+y} + y = 13;$$

and multiplying by $x+y$, and transposing,

$$(x+y)^2 - 13(x+y) = -36;$$

$\therefore$ completing the square (art. 111),

$$4(x+y)^2 - 52(x+y) + 169 = 25;$$

and extracting the root,

$$2(x+y) - 13 = \pm 5;$$

$$\therefore x + y = 9, \text{ or } 4;$$

whence $(x+y)^2 = 81$, or 16;

and subtracting $\quad 4xy = 72$,

we have $\quad (x-y)^2 = 9$, or -56;

$$\therefore x - y = 3, \text{ or } \pm\sqrt{-56} = \pm 2\sqrt{-14};$$

and adding, $\quad x + y = 9$,

$$2x = 12; \therefore x = 6;$$

also by subtracting $2y = 6$; $\therefore y = 3$;

hence the three numbers are 6, 4, and 3.

Otherwise.

Let the extremes be $x+y$, and $x-y$;

then the mean will be $\frac{x^2-y^2}{x}$;

and their sum $=2x+\frac{x^2-y^2}{x}=13$;

also the product of the extremes $=x^2-y^2=18$;

$\therefore$ by substitution,

$$2x+\frac{18}{x}=13;$$

and multiplying by x, and transposing,

$$2x^2-13x=-18;$$

$\therefore$ completing the square (art. 111),

$$16x^2-104x+169=25;$$

and extracting the root,

$$4x-13=\pm 5;$$

$$\therefore x=\tfrac{9}{2}, \text{ or } 2;$$

and substituting the first of these values in the equation $x^2-y^2=18$, we have

$$\tfrac{81}{4}-y^2=18, \therefore y^2=\tfrac{9}{4}, \text{ and } y=\tfrac{3}{2};$$

hence the numbers are 6, 4, and 3, as above.

QUESTION V.

It is required to find four numbers in arithmetical progression, such that the product of the extremes shall be 45, and the product of the means 77.

Let x be the first term, and y the common difference, then the numbers will be

$$x,\ x+y,\ x+2y,\ x+3y;$$

and by the question,

$$x^2 + 3xy \qquad = 45,$$
$$x^2 + 3xy + 2y^2 = 77\,;$$

$\therefore$ by subtraction . . . . . $2y^2 = 32,$

$$\therefore\ y = 4\,;$$

hence the first equation becomes

$$x^2 + 12x = 45\,;$$

which, solved, gives $x = 3$;

hence the four numbers are 3, 7, 11, and 15.

QUESTION VI.

It is required to find two numbers, such that their sum, product, and the difference of their squares may be equal to each other.

This question has already been solved, with only one unknown quantity (page 148): the solution with two unknown quantities is as follows:

Let x represent the greater number, and y the less.

then, by the question, $\begin{cases} x + y = xy \\ x + y = x^2 - y^2. \end{cases}$

Dividing each member of the second equation by $x + y$, we have

$$1 = x - y, \therefore\ y = x - 1.$$

Substituting this value of y in the first equation,

$$2x - 1 = x^2 - x,$$
$$\therefore\ x^2 - 3x = -1,$$

which, solved, gives

$$x = \frac{3 \pm \sqrt{5}}{2}, \therefore\ y = \frac{1 \pm \sqrt{5}}{2}.$$

7. There are two numbers, whose sum, multiplied by the greater, gives 144, and whose difference, multiplied by the less, gives 14. What are the numbers? Ans. 9 and 7, or $\sqrt{2}$ and $8\sqrt{2}$.

8. What number is that, which, being divided by the product of its two digits, the quotient is 2, and if 27 be added to the number, the digits will be inverted? Ans. 36.

9. A grocer sold 80 pounds of mace and 100 pounds of cloves for 65*l.*, and finds that he has sold 60 pounds more of cloves for 20*l.* than of mace for 10*l.* What was the price of a pound of each?

Ans. 1 lb. of mace is 10*s.*, and 1 lb. of cloves, 5*s.*

10. It is required to find three integral numbers whose sum is 38, such that the difference of the first and second shall exceed the difference of the second and third by 7, and the sum of whose squares is 634. Ans. 3, 15, and 20.*

11. There are three numbers in geometrical progression, whose sum is 52, and the sum of the extremes is to the mean as 10 to 3. What are the numbers? Ans. 4, 12, and 36.

12. It is required to find two numbers, such that their product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

Ans. $\frac{\sqrt{5}}{2}$, and $\frac{5+\sqrt{5}}{4}$.

13. The product of five numbers in arithmetical progression is 10395, and their sum is 35. What are the numbers?

Ans. 11, 9, 7, 5, and 3.

14. The sum of three numbers in geometrical progression is 13, and the product of the mean and sum of the extremes is 30. What are the numbers? Ans. 1, 3, and 9.

15. The arithmetical mean of two numbers exceeds the geometrical mean by 13, and the geometrical mean exceeds the harmonical mean by 12. Required the numbers. Ans. 234 and 104.

16. There are three numbers, the difference of whose differences is 5; their sum is 20, and their product 130. What are the numbers?

Ans. 2, 5, and 13.

17. The fore-wheel of a carriage makes six revolutions more than the hind-wheel in going 120 yards; but if the circumference of each

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* If fractional numbers be admitted, the following will also answer: $22\frac{1}{3}$, $10\frac{1}{3}$, and $5\frac{1}{3}$.

wheel be increased one yard, it will make only four revolutions more than the hind-wheel in going the same distance. Required the circumference of each.

Ans. The circumference of the fore-wheel is 4 yards; and of the hind-wheel 5 yards.

18. Find five numbers in geometrical progression, such that their sum may be 31, and the sum of their squares 341.*

Ans. 1, 2, 4, 8, and 16.

19. The sum of four numbers in geometrical progression is 30, and it is found that three times the last term is equal to four times the sum of the mean terms. Required the numbers.

Ans. 2, 4, 8, and 16.

20. Find three numbers in geometrical progression, such that their product may be 64, and the sum of their cubes 584.

Ans. 2, 4, and 8.

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* It is an interesting property of a geometrical progression, that the sum of any *odd* number of terms will always exactly divide the sum of their squares. Thus, let S be the sum, and S′ the sum of their squares; then, (page 99.)

$$S = \frac{a(r^n - 1)}{r - 1},\ S' = \frac{a^2(r^{2n} - 1)}{r^2 - 1}$$

$$\therefore \frac{S'}{S} = \frac{a(r^{2n} - 1)}{(r+1)(r^n - 1)} = \frac{a(r^n + 1)}{r + 1}$$

and (page 117) this, when n is *odd*, is

$$a - ar + ar^2 - ar^3 + \ldots\ ar^{n-1}, \ldots \quad [1]$$

which is the original series S with the alternate signs changed, and therefore expresses the sum of the odd terms diminished by the sum of the even terms.

It may be added that the preceding conclusion might have been deduced without reference to page 117, for the expression for $\frac{S'}{S}$ is the same as $\frac{a(-r^n - 1)}{(-r - 1)}$, which is obviously the expression for the sum of the series [1], the ratio r being negative, and n odd.

CHAPTER VI.

ON THE BINOMIAL THEOREM.

(124.) The Binomial Theorem is a theorem discovered by Sir Isaac Newton,* whereby any power or root of a binomial may be obtained without the labour of performing the actual involution or extraction. The power or root so found is usually called the *expansion* or *development* of the binomial; but, before we proceed to investigate the form or law of this development, it will be necessary to prove the truth of the following

THEOREM.

(125.) If the series $A + Bx + Cx^2 + Dx^3 + \&c.$, whether finite or infinite, be equal to the series $A' + B'x + C'x^2 + D'x^3 + \&c.$, whatever be the value of x, then the coefficients of the *like* powers of x will be respectively equal the one to the other; that is,

$$A = A', \; B = B', \; C = C', \; \&c.$$

For since the two series are equal, whatever be the value of x, they are equal when $x = 0$; but, in this case, the first series be-

* It appears to be not strictly correct to ascribe the first discovery of this celebrated theorem to Sir Isaac Newton, as it was known and applied in the case of integral powers before his time. (See Dr. Hutton's History of Logarithms.) Newton, however, was undoubtedly the first discoverer of the theorem under its present form, since none of his predecessors had ever shown its application in the cases of fractional or negative exponents. It is remarkable that, although this theorem was one of Newton's earliest discoveries, he has left no demonstration of it; and he is therefore supposed to have inferred its generality by induction from particular cases. (See Brewster's or Biot's Life of Newton.)

comes simply A, and the second becomes A′; therefore, A = A′: and it also follows, that

$$Bx + Cx^2 + Dx^3 + \&c. = B'x + C'x^2 + D'x^3 + \&c.;$$

whatever be the value of x: whence, dividing by x, and supposing $x = 0$, we have B = B′; and, by proceeding in a similar manner, it may be shown that C = C′, D = D′, &c.

It must be observed, however, that, after proving A = A′, if x were limited to the value $x = 0$, the remaining part of the proof would not follow; for the division of each member of the last equation by x would in that case introduce the forms $\frac{0}{0} = \frac{0}{0}$, the first of which is not necessarily B, nor the second necessarily B′. Each may, on the contrary, be any value whatever,; but as x, so far from being limited to the single value 0, is a perfectly general symbol, altogether unlimited, the result of the division spoken of can be no other than

$$B + Cx + Dx^2 + \&c. = B' + C'x + D'x^2 + \&c.$$

As this fundamental theorem is useful, not only in establishing the development of a binomial, but also in a great variety of other important investigations, we shall present to the student another method of proving the truth of it.

By transposition we have the equation

$$(A - A') + (B - B')x + (C - C')x^2 + (D - D')x^3 + \&c. = 0 \ldots [1],$$

whatever be the value of x. But when $x = 0$, this reduces to

$$A - A' = 0, \therefore A = A';$$

consequently, as the first term in the equation [1] vanishes, that equation is the same as

$$\{(B - B') + (C - C')x + (D - D')x^2 + \&c.\}\, x = 0.$$

But when the product of two factors is 0, one of those factors must necessarily be 0. The second factor x is not necessarily 0, but is, on the contrary, quite unrestricted; hence, the other factor is necessarily 0; that is,

$$(B - B') + (C - C')x + (D - D')x^2 + \&c. = 0 \ldots [2],$$

whatever be the value of x. Hence, as at first,

$$B - B' = 0, \therefore B = B';$$

so that the last equation [2] becomes

$$\{(C - C') + (D - D')x + \&c.\}\, x = 0,$$

and, as x is not necessarily 0,

$$(C - C') + (D - D')x + \&c. = 0,$$

and therefore, as at first,

$$C - C' = 0, \therefore C = C',$$

and so on.*

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* It has sometimes been objected to reasoning like that employed in the text, that although the terms after the first in the series [1] may be made individually as small as we please, by taking a sufficiently small value of x, yet when the number of these terms is infinite, their aggregate *may* be a value indefinitely great, however small x be taken. But that this cannot be the case, in any series proceeding according to the positive integral powers of x with finite coefficients, may be easily proved:

Let $a + bx + cx^2 + dx^3 + \&c.$
be any such series, and let k be that coefficient which is the greatest. Then, assuming all the terms after a to be plus,—the most unfavorable assumption for our purpose,—the sum of these terms cannot *exceed*

$$kx + kx^2 + kx^3 + \&c.$$

$$\text{or } kx\,(1 + x + x^2 + \&c.)$$

that is, summing the infinite geometrical series, the sum cannot exceed

$$kx \frac{1}{1 - x},$$

which approaches nearer and nearer to 0 as x diminishes, and actually attains it when x does, as affirmed in the text.

It may be proper to add here, that the theorem to which the present note refers admits of proof under a limitation of the data; it may be shown, by aid of the general theory of equations, that the identity of the coefficients of the like powers of x equally follows, provided the equation holds for all values of x between any given limits, as from $x = a$ up to $x = b$.

Investigation of the Binomial Theorem.

Let m be any number, either integral or fractional, positive or negative. It is required to exhibit the development of $(a + x)^m$, or of its equal $a^m (1 + \frac{x}{a})^m$, or, putting z for $\frac{x}{a}$, of $a^m (1 + z)^m$.

1. When m is a positive integer, the first term of the power $(1 + z)^m$ is obviously 1; and as to the other terms, it is plain that no fractional power of z can occur among them, for it is not in the nature of multiplication to *introduce* fractional powers. Every positive integral power of $(1 + z)$ may therefore be exhibited in a series of terms commencing with 1, and proceeding according to the *increasing integral powers of z*; that is, the *form* of the development in this case will be

$$1 + \mathrm{P}z + \mathrm{Q}z^2 + \mathrm{R}z^3 + \&\text{c.} \ldots . [1].$$

But what the numerical values of the coefficients P, Q, R, &c. may be for any proposed value of m, or whether some of them may not be zero, are inquiries to be entered upon hereafter.

2. When m is a positive fraction as $\frac{p}{q}$, then the required development will be the qth root of the pth power of $(1 + z)$, that is, from what is shown above, the qth root of an expression of the form [1]. Whatever this root may be, it is plain that the leading term of it must be 1, since it is from the involution of the root that [1] must be reproduced. It is easily seen, too, that into the subsequent terms of this root no fractional power of z can enter; for be these subsequent terms what they may, if they be only arranged according to the ascending powers of z, like [1] above, the *lowest* fractional power of z will continually recur at every step of the involution, occupying the same place from the leading term that it occupies in the root itself. This follows from the very nature of multiplication. For instance, whatever power we take of $1 + ax + cx^{\frac{5}{2}} + dx^3 + \&\text{c.}$, the fractional power $x^{\frac{5}{2}}$ will always reappear in the result, and at the same distance from the leading term of that result that it now is from the leading term, 1,

of the expression itself. For, filling up the vacant place in that expression, that is, writing it thus,

$$1 + ax + 0x^2 + cx^{\frac{5}{2}} + dx^3 + \&c.$$

we see that $x^{\frac{5}{2}}$ occupies the *fourth* place, and that it must reappear and continue to occupy the fourth place in the square, in the cube, and so on. We infer, therefore, that the lowest fractional power of z, supposing such to enter into the qth root of $(1 + z)^p$, must reappear in the qth power of that root, that is in $(1 + z)^p$ itself; for it can never happen that this fractional power can be neutralized by the entrance into the result of a like power with opposite sign, since no other fractional power so small as this enters into the process. But it has been shown that no fractional power of z can appear in $(1 + z)^p$. Hence, no fractional power can appear in $(1 + z)^{\frac{p}{q}}$; that is to say, that when m is a positive fraction, instead of a positive integer, the development is still of the form [1].

3. Lastly, let m be negative, and either integral or fractional; then, from what is proved above, the development of $(1 + z)^{-m}$, or, which is the same thing, of $\dfrac{1}{(1 + z)^m}$ will be no other than the quotient of 1, by an expression of the form [1]. The leading term of this quotient will therefore be 1; and as the first remainder can contain no fractional power of z, neither can the next term in the quotient contain such power; hence, the subsequent remainder can contain no such power, nor as a consequence, the next term in the quotient, and so on. Hence, generally, whether m be integral or fractional, positive or negative, the *form* of the development will always be

$$(1 + z)^m = 1 + \text{P}z + \text{Q}z^2 + \text{R}z^3 + \&c. \quad . \; . \; . \; [2].$$

It must be admitted, however, that this conclusion rigorously follows from the preceding reasoning only on the supposition that no *negative* power of z can ever enter into the development. The accuracy of this supposition is pretty evident; yet, to remove all ground of objection to the important inference just obtained, we

shall prove that no negative exponent can enter the development of $(1+z)^m$, as follows:

If negative exponents be possible, let the largest that enters be $-k$: then if every term in the development, after the leading term 1, be divided by z^{-k}, no negative powers of z can occur in the quotients, since this division requires the addition of k to each of the other exponents, of which those that are negative are every one of them numerically *less* than k. When negative exponents enter, therefore, the development may be written

$$(1+z)^m = 1 + \{a + bz^\alpha + cz^\beta + dz^\gamma + \&c.\} z^{-k} \; . \; . \; . \; [3]$$

in which the exponents α, β, γ, &c. are all *positive*. Put $z=0$ in the left-hand member of this equation, and that member becomes simply 1^m or 1. Put the same value of z in the other member; then all the terms within the brace vanish, with the exception of a; so that that member becomes $1+\frac{a}{0}$, or *infinite.** It is impossible therefore that the presumed equality can exist; that is, no negative power of z can enter the development.

(126.) Having thus established the form of the development, as far at least as regards the powers of z, let us inquire into the nature of the coefficients P, Q, R, &c.

And first, it may be remarked, that when these coefficients are actually known for any given value n of the exponent, their values for the exponent $n+1$ will be soon ascertained by multiplying the known development by $1+z$. Thus, if

$$(1+z)^n = 1 + pz + qz^2 + \&c. \; . \; . \; . \; . \; . \; . \; [\text{A}],$$

then, multiplying by $1 + z$

$$\begin{array}{l} 1 + pz + qz^2 + \&c. \\ \quad\;\; z + pz^2 + \&c. \end{array}$$

we have $(1+z)^{n+1} = 1 + (p+1)z + (p+q)z^2 + \&c. \; . \; . \; . \; [\text{B}]$.

* To some readers, the impossibility of the equation [3] may seem more clearly established, thus: multiply each side of that equation by z^k, put $z=0$, in the result, and we have $0=a$, that is, the coefficient of the supposed negative power is *nothing*, so that such a power cannot appear.

The difference between the exponent n and the coefficient p of the second term in the corresponding development [A] is $n - p$; and we now see that although we increase that exponent by 1, yet the difference between this new exponent, and the coefficient of the second term in the new development [B] corresponding, remains unaltered; for $(n + 1) - (p + 1)$ is still $n - p$. Now when $n = 1$ we have $(1 + z)^1 = 1 + z$, and as the coefficient of the second term is in this case 1, as well as the exponent itself, the difference between them is 0; hence, from what is shown above, the difference between the exponent and coefficient of z in the development is 0 when $n = 2$, and therefore also 0 when $n = 3$, when $n = 4$, and generally when n is any positive whole number whatever.

It follows therefore that, when the exponent is any positive integer, the coefficient of the second term of the development will always be equal to that exponent: that is,

$$(1 + z)^m = 1 + mz + Qz^2 + Rz^3 + \&c. \quad . \; . \; . \; [4].$$

We have thus ascertained the value of P when the exponent is a positive whole number:—it is that exponent itself.

The same relationship between the exponent and coefficient of the second term of the development exists also when the exponent is negative. For

$$(1 + z)^{-m} = \frac{1}{1 + mz + Qz^2 + \&c.};$$

and, by actual division, the first two terms of the quotient is $1 - mz$, showing that, whatever the subsequent coefficients may be, that of z is, as before, equal to the exponent $-m$.

It remains to be ascertained whether the like equality continues even when the exponent is fractional. In order to this, let m be $\pm \frac{p}{q}$; then [2]

$$(1 + z)^{\pm \frac{p}{q}} = 1 + Pz + Qz^2 + \&c. \quad . \; . \; . \; [5],$$

or, putting for brevity z' for $Pz + Qz^2 + \&c.$

$$(1 + z)^{\pm \frac{p}{q}} = 1 + z'$$
$$\therefore (1 + z)^{\pm p} = (1 + z')^q$$

and substituting for these their developments as far as the second term, we have, from what has just been proved,

$$1 \pm pz + \&c. = 1 + qz' + \&c.$$

that is, replacing z' by its value,

$$1 \pm pz + \&c. = 1 + q(\text{P}z + \text{Q}z^2 + \&c.) + \&c.$$

whatever be the value of z. Hence, by equating the coefficients of z (125), we have

$$\pm p = q\text{P}, \therefore \pm \frac{p}{q} = \text{P},$$

that is, the coefficient of the second term of the development [5] is also in this case equal to the exponent $\pm \frac{p}{q}$.

We have thus proved that, whatever number m may be, we shall always have

$$(1 + z)^m = 1 + mz + \text{Q}z^2 + \text{R}z^3 + \&c. \quad . \quad . \quad . \quad [6],$$

in which however the coefficients Q, R, &c. are still undetermined. If then we replace z by its value $\frac{x}{a}$ we may be certain that, as far as two terms, the development of the proposed binomial is

$$a^m \left(1 + \frac{x}{a}\right)^m = a^m + ma^m \frac{x}{a} + \&c.$$

that is,

$$(a + x)^m = a^m + ma^{m-1} x + \&c. \quad . \quad . \quad . \quad [7],$$

and we have now to determine the constitution of the subsequent coefficients.

For this purpose put, in [6], $b(1 + x)$ for z, which is allowable, since z is altogether arbitrary. We shall then have

$$\{1 + b(1 + x)\}^m = 1 + mb(1 + x) + \text{Q}b^2(1 + x)^2 + \text{R}b^3(1 + x)^3$$
$$+ \text{S}b^4(1 + x)^4 + \&c.$$
$$= 1 + mb + \text{Q}b^2 + \text{R}b^3 + \text{S}b^4 + \&c.$$
$$+ (mb + 2\text{Q}b^2 + 3\text{R}b^3 + 4\text{S}b^4 + \&c.)\, x + \&c.$$

But $1 + b(1 + x)$ is the same as $(1 + b) + bx$; and the development of this, as far as two terms, will be obtained from [7] by

substituting, in that development, $1 + b$ for a, and bx for x. We thus get

$$\{(1 + b) + bx\}^m = (1 + b)^m + m(1 + b)^{m-1} bx + \&c.$$

As this development must be equal to the former, whatever be the value of x, we must have, by equating the coefficients of x, (125),

$$m(1 + b)^{m-1} b = mb + 2Qb^2 + 3Rb^3 + 4Sb^4 + \&c.$$

or

$$m(1 + b)^{m-1} = m + 2Qb + 3Rb^2 + 4Sb^3 + \&c.$$

Now by [6]

$$(1 + b)^m = 1 + mb + Qb^2 + Rb^3 + Sb^4 + \&c.$$

Hence, multiplying this by m and the equation just deduced by $(1 + b)$, the two will become identical; so that

$$m + m^2b + mQb^2 + mRb^3 + \&c.$$
$$= m + (2Q + m)b + (3R + 2Q)b^2 + (4S + 3R)b^3 + \&c.$$

Consequently, equating the coefficients of the like terms,

$$2Q + m = m^2 \therefore Q = \frac{m(m-1)}{2}$$

$$3R + 2Q = mQ \therefore R = \frac{m(m-1)(m-2)}{2.3}$$

$$4S + 3R = mR \therefore S = \frac{m(m-1)(m-2)(m-3)}{2.3.4} \quad \&c. \ \&c.$$

And thus the law of the coefficients Q, R, S, &c. becomes known; so that, by [6], whatever be m, we shall always have

$$(1 + \frac{x}{a})^m = 1 + m\frac{x}{a} + \frac{m(m-1)}{2}(\frac{x}{a})^2 + \frac{m(m-1)(m-2)}{2.3}(\frac{x}{a})^3 + \&c. \ldots. [\text{I}],$$

or, multiplying by a^m,

$$(a + x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{2}a^{m-2}x^2 + \frac{m(m-1)(m-2)}{2.3}a^{m-3}x^3 + \&c. \ldots. [\text{II}];$$

and this is the *Binomial Theorem.*

The development thus exhibited may be easily recalled at any time, merely from remembering the uniform law which connects each term with the immediately preceding one. Thus the first term of the development of $(a+x)^m$ is a^m, and (omitting the coefficients) every subsequent term is formed by multiplying that which immediately precedes it by $\frac{x}{a}$, so that the powers of a regularly decrease by 1, whilst those of x as regularly increase by 1.

As to the coefficients, that of the second term is m; and of the others, the numerator of each is equal to the numerator of the preceding multiplied by an additional factor; and the denominator equal to the denominator of the preceding, multiplied by an additional factor; the factors in the numerator uniformly decreasing by 1, those of the denominator uniformly increasing by 1.

It appears from [II] that the additional factor which thus enters into the numerator of every new term is no other than the exponent of a in the preceding term; and as the additional factor in the denominator is the exponent of x in the same term, increased by 1, it follows that, when any term of the development [II] is given with its coefficient and exponents expressed in *figures*, the immediately succeeding term, and thence the following terms, may be readily written down without inquiring from what value of m the given numerical coefficient has arisen. Thus, if $56a^5x^3$ be a term in the development for a certain numerical value of m, then the next term will have for coefficient $\frac{56 \times 5}{4} = 70$; and, consequently, the complete term will be $70a^4x^4$.

From attending to the formation of the coefficients, it is clear that the development will terminate only when m is integral and positive. For it is only then that the continued subtraction of unity from m, in order to supply the several successive factors, can exhaust m, and thus put a stop to the series. When, therefore, the exponent is either negative or fractional, the development is an infinite series.

It is worthy of remark that in the former case, that is, when the development is finite, the coefficients of any two terms equidistant from the extreme terms are always equal; so that, in

writing down the development due to any positive integral value of m, we need compute the coefficients only up to the *middle* of the series, and then repeat them, in reverse order, to the end. Whereabouts the middle of the series is will always be discoverable from the known value of m. For, referring to the general development, it is evident that the coefficients continue to succeed each other so long as the successive abstractions of 1, 2, 3, 4, &c. from m continue to leave a remainder; that is, till the subtractive number becomes $m - 1$, which is evidently at the $m - 1$th term beyond the second; hence the entire number of terms is $m + 1$, so that, if m be even, the $\frac{1}{2}m + 1$th is the middle term, and there are $\frac{1}{2}m$ terms on each side of it; but if m be odd, the $\frac{m+1}{2}$th, and next following term, occupy the middle, and have $\frac{m-1}{2}$ terms on each side of them.

That the coefficients are really equal at equal distances from the extreme terms, will be seen from considering that whether we develop $(a + x)^m$ or $(x + a)^m$, the same coefficients must appear, and in the same order; but the literal quantities to which they are prefixed do not occur in the same order in both series, but, on the contrary, in reverse order. For, attending to the law of the exponents, the first, second, third, &c. terms of $(a + x)^m$ (omitting the coefficients) are

$$a^m x^0,\ a^{m-1}x^1,\ a^{m-2}x^2,\ \&c.,$$

and the last, last but one, last but two, &c. of $(x + a)^m$ are

$$x^0 a^m,\ x^1 a^{m-1},\ x^2 a^{m-2},\ \&c.$$

that is, proceeding along the development of $(x + a)^m$ from the last term to the first, we meet with the same succession of literal quantities that we do in proceeding from the first term to the last in the development of $(a + x)^m$; so that, if we take any term in one series, and note its distance from the *beginning*, and then a term in the other series at an equal distance from the *end*, we shall find these terms to be *like*, or to involve the same letters. Hence, as the two series are always *equal*, whatever be the value

of x, the coefficients of the *like* terms are equal (121); that is the coefficient of the leading term in one is equal to that of the final term in the other; the coefficient of the second term in the one equal to that of the last but one in the other, and so on. But, as remarked above, the coefficients in the two developments $(a+x)^m$, $(x+a)^m$ are the same, and occur in the same order; hence, instead of saying that the leading coefficient in the one is the same as the final coefficient in the *other*, the second in the one the same as the last but one in the *other*, and so on, we may say that the first and last terms in *either* series must have equal coefficients, the second and last but one, the third and last but two, and so on.

It is plain that for integral exponents the coefficients are all integral: for, from the nature of multiplication, the development of $(a+x)^n$, n being a positive integer, cannot possibly involve fractions; and from the nature of division, the development of

$$(a+x)^{-n} \text{ or } \frac{1}{a^n + na^{n-1}x + \&c.}$$

is equally free from fractions, seeing that the leading coefficient of the divisor is *unity*, and that the other coefficients are all integral. It thus follows that the product of n consecutive whole numbers as always divisible by the product $1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots n$.

(127.) We shall add but one more remark, viz. that, by making a and x each equal to 1, we get the following curious property of the binomial, viz.

$$(1+1)^m, \text{ or } 2^m, = 1 + m + \frac{m(m-1)}{2} +$$

$$\frac{m(m-1)(m-2)}{2\,.\,3} + \frac{m(m-1)(m-2)(m-3)}{2\,.\,3\,.\,4} + \&c.$$

that is, in any development of a binomial, whose terms are both positive, the sum of the coefficients is equal to the same power, or root, of 2.

Also, if $a = 1$, $x = -1$, and m positive, we have the following property, viz.

$$(1-1)^m, \text{ or } 0, = 1 - m + \frac{m(m-1)}{2} - \frac{m(m-1)(m-2)}{2.3} + \frac{m(m-1)(m-2)(m-3)}{2.3.4} - \&c.$$

that is, in the development of any positive power of a binomial, one of whose terms is negative, the sum of the coefficients is $= 0$; and therefore the sum of the positive coefficients must be equal to the sum of the negative ones. Hence, in every such development the sum of the first, third, fifth, &c. coefficients is equal to the sum of the second, fourth, sixth, &c. coefficients; therefore each sum is equal to half the entire series, that is, to $\frac{2^m}{2}$ or 2^{m-1}.

On account of the great importance of the Binomial Theorem in almost every department of Analysis, we feel disposed to present the student with another method of establishing the law of the coefficients. But, not to detain him longer from its practical application, we shall postpone this second method of investigation till the close of the Chapter (see page 206).

APPLICATIONS OF THE BINOMIAL THEOREM.

(128.) 1. *To develop $(a + x)^m$ when m is a Positive or Negative Integer.*

Make the first and second terms a^m, and $ma^{m-1}x$, respectively; then, to find the others, multiply the coefficient of the term last found by the index of a in that term; and the product divided by the number of that term, or by the exponent of x in it, increased by 1, will give the coefficient of the next term:* with respect to

* It is worthy of notice that one or other of the two factors forming the product will always be exactly divisible by the number of the term, whenever that number is not itself decomposable into factors; that is,

the literal parts, the powers of a are to *decrease* and those of x to *increase* by unity in each successive term.

NOTE. When m is positive, the coefficients need only be calculated as far as the middle, those of the other terms being the same, taken in an inverted order (page 192). If one part of the binomial be *negative*, then the terms involving its odd powers must evidently be negative.

EXAMPLES.

1. It is required to develop $(a+x)^8$, or to express the 8th power of $(a+x)$.

Here

The first term is . . . . a^8.

The second . . . . $8a^7x$.

The third . . . . $\frac{8 \times 7}{2} a^6x^2 = 28a^6x^2$.

The fourth . . . . $\frac{28 \times 6}{3} a^5x^3 = 56a^5x^3$.

The fifth . . . . $\frac{56 \times 5}{4} a^4x^4 = 70a^4x^4$.

And from these the other terms are obtained (NOTE).

Hence $(a+x)^8 = a^8 + 8a^7x + 28a^6x^2 + 56a^5x^3 + 70a^4x^4 + 56a^3x^5 + 28a^2x^6 + 8ax^7 + x^8$.

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when it is one of the numbers 2, 3, 5, 7, 11, 13, &c.; and the same will often happen when the number marking the place of the term *is* decomposable into factors; when it does not happen, the two component factors will, of course, always divide those of the aforesaid product (page 193); and from these considerations the numerical labour of development may be materially reduced.

2. It is required to develop $(x-y)^9$, or to find the 9th power of $(x-y)$.

Here

The first term is . . . . x^9.

The second . . . . $-9x^8y$.

The third . . . . $\frac{9\times 8}{2}x^7y^2 = 36x^7y^2$.

The fourth . . . . $-\frac{36\times 7}{3}x^6y^3 = -84x^6y^3$.

The fifth . . . . $\frac{84\times 6}{4}x^5y^4 = 126x^5y^4$.

&c. &c, &c.

Hence $(x-y)^9 = x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$.

3. It is required to develop $(a+x)^{-2}$, or $\frac{1}{(a+x)^2}$.

Here

The first term is . . . . a^{-2}.

The second . . . . $-2a^{-3}x$.

The third . . . . $\frac{-2\times -3}{2}a^{-4}x^2 = 3a^{-4}x^2$.

The fourth . . . . $\frac{3\times -4}{3}a^{-5}x^3 = -4a^{-5}x^3$.

&c. &c. &c.

Hence

$$(a+x)^{-2}, \text{or } \frac{1}{(a+x)^2}, = a^{-2} - 2a^{-3}x + 3a^{-4}x^2 - 4a^{-5}x^3 + \&c. =$$

$$\frac{1}{a^2} - \frac{2x}{a^3} + \frac{3x^2}{a^4} - \frac{4x^3}{a^5} + \&c. = \frac{1}{a^2}(1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \&c.)$$

4. It is required to develop $(a+2x)^{-3}$, or $\dfrac{1}{(a+2x)^3}$.

Here

The first term is . . . . a^{-3}.

The second . . . . $-3a^{-4}2x = -6a^{-4}x$.

The third . . . . $\dfrac{-3\times-4}{2}a^{-5}(2x)^2 = 24a^{-5}x^2$.

The fourth . . . . . . $\dfrac{6\times-5}{3}a^{-6}(2x)^3 = -80a^{-6}x^3$.

&c. &c. &c.

Hence

$$(a+2x)^{-3}, \text{ or } \frac{1}{(a+2x)^3} = a^{-3} - 6a^{-4}x + 24a^{-5}x^2 - 80a^{-6}x^3 + \&c.$$

$$= \frac{1}{a^3} - \frac{6x}{a^4} + \frac{24x^2}{a^5} - \frac{80x^3}{a^6} + \&c. = \frac{1}{a^3}\left(1 - \frac{6x}{a} + \frac{24x^2}{a^2} - \frac{80x^3}{a^3} + \&c.\right)$$

5. It is required to develop $(x-y)^7$, or to find the 7th power of $(x-y)$.

Ans. $\begin{cases} x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + \\ 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7. \end{cases}$

6. It is required to find the 7th power of $(x+2y)$.

Ans. $\begin{cases} x^7 + 14x^6y + 84x^5y^2 + 280x^4y^3 + 560x^3y^4 + \\ 672x^2y^5 + 448xy^6 + 128y^7. \end{cases}$

7. It is required to find the cube of $a+b+c$, or the development of $[(a+b)+c]^3$.

Ans. $(a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$,
or $a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

8. It is required to find the development of $\dfrac{2}{(c+x)^2}$.

Ans. $\dfrac{2}{c^2} - \dfrac{4x}{c^3} + \dfrac{6x^2}{c^4} - \dfrac{8x^3}{c^5} + \&c.$

9. It is required to find the development of $\dfrac{a^2}{(a+2b)^3}$.

Ans. $\dfrac{1}{a}\left(1 - \dfrac{6b}{a} + \dfrac{24b^2}{a^2} - \dfrac{80b^3}{a^3} + \&c.\right)$

(129.) 2. *To develop* $(a+x)^{\frac{m}{n}}$, $\frac{m}{n}$ *being either Positive or Negative.*

Agreeably to the law of the terms already established (p. 190), if we put Q for $\frac{x}{a}$, we shall have

$$(a+x)^{\frac{m}{n}} = \overbrace{a^{\frac{m}{n}}}^{\text{A}} + \overbrace{\frac{m}{n}\text{AQ}}^{\text{B}} + \overbrace{\frac{\frac{m}{n}-1}{2}\text{BQ}}^{\text{C}} + \overbrace{\frac{\frac{m}{n}-2}{3}\text{CQ}}^{\text{D}} + \&\text{c.}$$

where A, B, C, &c. represent the first, second, third, &c. terms respectively.

Or,

$$(a+x)^{\frac{m}{n}} = \overbrace{a^{\frac{m}{n}}}^{\text{A}} + \overbrace{\frac{m}{n}\text{AQ}}^{\text{B}} + \overbrace{\frac{m-n}{2n}\text{BQ}}^{\text{C}} + \overbrace{\frac{m-2n}{3n}\text{CQ}}^{\text{D}} + \&\text{c.}$$

which last form is the most commodious in practice, and differs but very little from that in which the binomial theorem was first proposed by Newton.*

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* It will be frequently found the most expeditious mode of development, first to write down, by the former rule, the series for $(a+x)$ raised to the given fractional power; and then underneath to put the proper substitutions for a and x. Even should the second term of the proposed binomial be negative, still our guiding development should be that of $(a+x)$, because, as the exponents alone determine the signs of the succeeding coefficients in $(a+x)^{\frac{m}{n}}$, they may be rapidly determined; and then, in our particular substitutions, it will be easy to take account of the sign of x.

According to the method here recommended, the work of example 4 following, will stand thus:

$$(a+x)^{\frac{3}{4}} = a^{\frac{3}{4}} + \frac{3}{4}a^{-\frac{1}{4}}x - \frac{3}{4\,.\,8}a^{-\frac{5}{4}}x^2 + \frac{3\,.\,5}{4\,.\,8\,.\,12}a^{-\frac{9}{4}}x^3 - \&\text{c.}$$

$$\therefore (b^2-x)^{\frac{3}{4}} = b^{\frac{3}{2}} - \frac{3}{4}b^{-\frac{1}{2}}x - \frac{3}{4\,.\,8}b^{-\frac{5}{2}}x^2 - \frac{3\,.\,5}{4\,.\,8\,.\,12}b^{-\frac{9}{2}}x^3 - \&\text{c.}$$

EXAMPLES.

1. Express the value of $\sqrt[3]{b^3+x}$ in a series.

Here $(a+x)^{\frac{m}{n}}=(b^3+x)^{\frac{1}{3}}$, $\therefore a=b^3$, $m=1$, $n=3$, and $Q=\frac{x}{b^3}$.

Whence $a^{\frac{m}{n}}=(b^3)^{\frac{1}{3}}=b=A.$

$$\frac{m}{n}AQ=\tfrac{1}{3}b\times\frac{x}{b^3}=\frac{x}{3b^2}=B.$$

$$\frac{m-n}{2n}BQ=-\tfrac{2}{6}\times\frac{x}{3b^2}\times\frac{x}{b^3}=-\frac{2x^2}{3\,.\,6b^5}=C.$$

$$\frac{m-2n}{3n}CQ=-\tfrac{5}{9}\times-\frac{2x^2}{3\,.\,6b^5}\times\frac{x}{b^3}=\frac{2\,.\,5x^3}{3\,.\,6\,.\,9b^8}=D.$$

$$\frac{m-3n}{4n}DQ=-\tfrac{8}{12}\times\frac{2\,.\,5x^3}{3\,.\,6\,.\,9b^8}\times\frac{x}{b^3}=-\frac{2\,.\,5\,.\,8x^4}{3\,.\,6\,.\,9\,.\,12b^{11}}=E.$$

Here the law of continuation is manifest;

$$\therefore\ \sqrt[3]{b^3+x}=b+\frac{x}{3b^2}-\frac{2x^2}{3\,.\,6b^5}+\frac{2\,.\,5x^3}{3\,.\,6\,.\,9b^8}-\frac{2\,.\,5\,.\,8x^4}{3\,.\,6\,.\,9\,.\,12b^{11}}+\&c.$$

2. Find the value of $\frac{1}{(b^2+x)^{\frac{1}{2}}}$ in a series.

Here $\frac{1}{(b^2+x)^{\frac{1}{2}}}=(b^2+x)^{-\frac{1}{2}}$, $\therefore a=b^2$, $m=-1$, $n=2$,

and $Q=\frac{x}{b^2}$.

Whence $a^{\frac{m}{n}}=(b^2)^{-\frac{1}{2}}=\frac{1}{b}=A.$

$$\frac{m}{n}AQ=-\frac{1}{2b}\times\frac{x}{b^2}=-\frac{x}{2b^3}=B.$$

$$\frac{m-n}{2n}BQ=-\tfrac{3}{4}\times-\frac{x}{2b^3}\times\frac{x}{b^2}=\frac{3x^2}{2\,.\,4b^5}=C.$$

$$\frac{m-2n}{3n}CQ=-\tfrac{5}{6}\times\frac{3x^2}{2\,.\,4b^5}\times\frac{x}{b^2}=-\frac{3\,.\,5x^3}{2\,.\,4\,.\,6b^7}=D.$$

$$\frac{m-3n}{4n}DQ=-\tfrac{7}{8}\times-\frac{3\,.\,5x^3}{2\,.\,4\,.\,6b^7}\times\frac{x}{b^2}=\frac{3\,.\,5\,.\,7x^4}{2\,.\,4\,.\,6\,.\,8b^9}=E.$$

&c. &c. &c.

$$\therefore\frac{1}{(b^2+x)^{\frac{1}{2}}}=\frac{1}{b}-\frac{x}{2b^3}+\frac{3x^2}{2\,.\,4b^5}-\frac{3\,.\,5x^3}{2\,.\,4\,.\,6b^7}+\frac{3\,.\,5\,.\,7x^4}{2\,.\,4\,.\,6\,.\,8b^9}-\&c.$$

3. Find the value of $(b^2 - x)^{\frac{1}{2}}$ in a series.

Here $a = b^2$, $x = -x$, $m = 1$, $n = 2$, and $Q = -\frac{x}{b^2}$.

Whence $a^{\frac{m}{n}} = (b^2)^{\frac{1}{2}} = b = A$.

$$\frac{m}{n}AQ = \tfrac{1}{2}b \times -\frac{x}{b^2} = -\frac{x}{2b} = B.$$

$$\frac{m-n}{2n}BQ = -\tfrac{1}{4} \times -\frac{x}{2b} \times -\frac{x}{b^2} = -\frac{x^2}{2.4b^3} = C.$$

$$\frac{m-2n}{3n}CQ = -\tfrac{3}{6} \times -\frac{x^2}{2.4b^3} \times -\frac{x}{b^2} = -\frac{3x^3}{2.4.6b^5} = D.$$

$$\frac{m-3n}{4n}DQ = -\tfrac{5}{8} \times -\frac{3x^3}{2.4.6b^5} \times -\frac{x}{b^2} = -\frac{3.5x^4}{2.4.6.8b^7} = E.$$

&c. &c. &c.

$$\therefore (b^2 - x)^{\frac{1}{2}} = b - \frac{x}{2b} - \frac{x^2}{2.4b^3} - \frac{3x^3}{2.4.6b^5} - \frac{3.5x^4}{2.4.6.8b^7} - \text{\&c.*}$$

4. Find the value of $(b^2 - x)^{\frac{3}{4}}$ in a series.

Here $a = b^2$, $x = -x$, $m = 3$, $n = 4$, and $Q = -\frac{x}{b^2}$.

Whence $a^{\frac{m}{n}} = (b^2)^{\frac{3}{4}} = b^{\frac{3}{2}} = A$.

$$\frac{m}{n}AQ = \tfrac{3}{4}b^{\frac{3}{2}} \times -\frac{x}{b^2} = -\frac{3x}{4b^{\frac{1}{2}}} = B.$$

$$\frac{m-n}{2n}BQ = -\tfrac{1}{8} \times -\frac{3x}{4b^{\frac{1}{2}}} \times -\frac{x}{b^2} = -\frac{3x^2}{4.8b^{\frac{5}{2}}} = C.$$

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* Finite values for a great variety of numerical infinite series may be readily obtained from different binomial developments: thus, in the present example, by putting $b^2 = x$, we have

$$0 = 1 - \frac{1}{2} - \frac{1}{2.4} - \frac{3}{2.4.6} - \frac{3.5}{2.4.6.8} - \text{\&c.}$$

$$\therefore \frac{1}{2} = \frac{1}{2.4} + \frac{3}{2.4.6} + \frac{3.5}{2.4.6.8} + \text{\&c.}$$

$$\frac{m-2n}{3n}\,\text{CQ} = -\tfrac{5}{12} \times -\frac{3x^2}{4\,.\,8b^{\frac{5}{2}}} \times -\frac{x}{b^2} = -\frac{3\,.\,5x^3}{4\,.\,8\,.\,12b^{\frac{9}{2}}} = \text{D}.$$

$$\frac{m-3n}{4n}\,\text{DQ} = -\tfrac{9}{16} \times -\frac{3\,.\,5x^3}{4\,.\,8\,.\,12b^{\frac{9}{2}}} \times -\frac{x}{b^2} = -\frac{3\,.\,5\,.\,9x^4}{4\,.\,8\,.\,12\,.\,16b^{\frac{13}{2}}} = \text{E}.$$

&c. &c. &c.

$$\therefore (b^2-x)^{\frac{3}{4}} = b^{\frac{3}{2}} - \frac{3x}{4b^{\frac{1}{2}}} - \frac{3x^2}{4\,.\,8b^{\frac{5}{2}}} - \frac{3\,.\,5x^3}{4\,.\,8\,.\,12b^{\frac{9}{2}}} - \frac{3\,.\,5\,.\,9x^4}{4\,.\,8\,.\,12\,.\,16b^{\frac{13}{2}}} -$$

&c. Or,

$$(b^2-x)^{\frac{3}{4}} = \frac{1}{\sqrt{b}}\left(b^2 - \frac{3x}{2^2} - \frac{3x^2}{2^5b^2} - \frac{5x^3}{2^7b^4} - \frac{5\,.\,9x^4}{2^{11}b^6} - \text{\&c.}\right)$$

5. Express the value of $\sqrt[3]{7}$ in a series.

Here $\sqrt[3]{7} = (8-1)^{\frac{1}{3}}$, $\therefore a = 8$, $x = -1$, $m = 1$, $n = 3$, and

$$\text{Q} = -\tfrac{1}{8} = -\frac{1}{2^3}.$$

Whence $a^{\frac{m}{n}} = 8^{\frac{1}{3}} = 2 = \text{A}$.

$$\frac{m}{n}\,\text{AQ} = \tfrac{1}{3} \times 2 \times -\frac{1}{2^3} = -\frac{1}{3\,.\,2^2} = \text{B}.$$

$$\frac{m-n}{2n}\,\text{BQ} = -\tfrac{2}{6} \times -\frac{1}{3\,.\,2^2} \times -\frac{1}{2^3} = -\frac{1}{3\,.\,6\,.\,2^4} = \text{C}.$$

$$\frac{m-2n}{3n}\,\text{CQ} = -\tfrac{5}{9} \times -\frac{1}{3\,.\,6\,.\,2^4} \times -\frac{1}{2^3} = -\frac{5}{3\,.\,6\,.\,9\,.\,2^7} = \text{D}.$$

$$\frac{m-3n}{4n}\,\text{DQ} = -\tfrac{8}{12} \times -\frac{5}{3\,.\,6\,.\,9\,.\,2^7} \times -\frac{1}{2^3} = -\frac{5\,.\,8}{3\,.\,6\,.\,9\,.\,12\,.\,2^{10}} = \text{E}.$$

&c. &c. &c.

$$\therefore \sqrt[3]{7} = 2 - \frac{1}{3\,.\,2^2} - \frac{1}{3\,.\,6\,.\,2^4} - \frac{5}{3\,.\,6\,.\,9\,.\,2^7} - \frac{5\,.\,8}{3\,.\,6\,.\,9\,.\,12\,.\,2^{10}} - \text{\&c.}$$

NOTE. The value of any irrational number may always be approximated to, by the binomial theorem, within any degree of nearness, as in the last example. But, as such approximate values

are usually required in decimals, it will in such cases be better to change the fractions into decimals at the outset. The rate of approximation, too, will generally be much increased by computing only a few leading terms of the series, and then recommencing with the sum of these for a new leading term, and so on, as in the following example.

6. Required the cube root of 9, to twelve or thirteen places of decimals.

Here $9^{\frac{1}{3}} = (8 + 1)^{\frac{1}{3}}$, $\therefore a = 8$, $x = 1$, $m = 1$, $n = 3$, and

$$Q = \tfrac{1}{8} = \cdot 125.$$

$$\text{Whence } a^{\frac{m}{n}} = 8^{\frac{1}{3}} \quad \ldots\ldots\ldots\ldots = 2 \qquad = A$$

$$\frac{m}{n} AQ = \tfrac{2}{3} \times \cdot 125 \quad \ldots\ldots\ldots = \quad \cdot 08333 = B$$

$$\frac{m-n}{2n} BQ = -\tfrac{1}{3} \times \tfrac{1}{8} \times \cdot 08333 \quad \ldots\ldots = - \cdot 00347 = C$$

$$\frac{m-2n}{3n} CQ = -\tfrac{5}{9} \times \tfrac{1}{8} \times - \cdot 00347 \quad \ldots\ldots = \quad \cdot 00024 = D$$

$$\frac{m-3n}{4n} DQ = -\tfrac{8}{12} \times \tfrac{1}{8} \times \cdot 0024 \ldots\ldots\ldots = - \cdot 00002 = E$$

$$\text{Approximate root} = \underline{2 \cdot 08008}$$

Again, $9^{\frac{1}{3}} = (2 \cdot 08008^3 + \cdot 000049624)^{\frac{1}{3}}$,

$$\therefore Q = \frac{\cdot 000049624}{2 \cdot 08008^3} = \cdot 000005513815l.$$

$$2 \cdot 08008 \qquad = A$$

$$\tfrac{1}{3} AQ = \quad \cdot 0000038230588 = B$$

$$-\tfrac{1}{3} BQ = - \cdot 0000000000069 = C$$

$$\text{Root to thirteen decimals} = \underline{2 \cdot 0800838230519}$$

7. Required the value of $\sqrt{b^2+x}$ in a series.

$$\text{Ans. } b+\frac{x}{2b}-\frac{x^2}{2.4b^3}+\frac{3x^3}{2.4.6b^5}-\frac{3.5x^4}{2.4.6.8b^7}+\&c.$$

8. Required the value of $\dfrac{c^2}{(c^2-x)^{\frac{1}{2}}}$ in a series.

$$\text{Ans. } c+\frac{x}{2c}+\frac{3x^2}{2.4c^3}+\frac{3.5x^3}{2.4.6c^5}+\frac{3.5.7x^4}{2.4.6.8c^7}+\&c.$$

9. Required the value of $(a+x)^{\frac{2}{3}}$ in a series.

$$\text{Ans } a^{\frac{2}{3}}\left(1+\frac{2x}{3a}-\frac{x^2}{3^2a^2}+\frac{4x^3}{3^4a^3}-\frac{7x^4}{3^5a^4}+\&c.\right)$$

10. Required the value of $\sqrt[3]{9}$ in a series.

$$\text{Ans. } 2+\frac{1}{3.2^2}-\frac{2}{3.6.2^4}+\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}+\&c.$$

11. Required the value of $\sqrt{2}$ in a series.

$$\text{Ans. } 1+\frac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}+\&c.$$

12. Required the value of $(a^2-x^2)^{\frac{3}{4}}$ in a series.

$$\text{Ans. } \frac{1}{\sqrt{a}}\left(a^2-\frac{3x^2}{2^2}-\frac{3x^4}{2^5a^2}-\frac{5x^6}{2^7a^4}-\frac{5.9x^8}{2^{11}a^6}-\&c.\right)$$

13. It is required to prove that every power or root of an expression of the form $a\pm b\sqrt{-1}$ is of the same form, viz., $P\pm Q\sqrt{-1}$.

SCHOLIUM.

The student will bear in mind that in all the foregoing examples the object has been to exhibit, in the form of a series whose terms succeed each other according to a general law, the results of certain operations performed upon binomial expressions. These results however, as we have already shown, cannot be *fully* exhibited

except the operation in question be one of simple involution; seeing that when the exponent of the binomial is either negative or fractional, the development never terminates. In either of these latter cases therefore, however great be the *finite* number of terms we may take as an equivalent to the whole, or to the undeveloped binomial, we necessarily commit an error amounting to the difference between the sum of the terms taken and the undeveloped expression; so that if, of the development of $(a + x)^m$, the portion taken be represented by s, the portion rejected must be equivalent to $(a + x)^m - s$; and that portion therefore is strictly the development of this last expression. Such a supplementary quantity should always be regarded as comprehended under the "&c." which follows the term at which we stop, in every practical application of the binomial theorem. In those applications which are purely arithmetical, our object will be to extend the terms comprehended under s sufficiently far to render the supplementary correction too insignificant to interfere with the degree of numerical accuracy we may wish to secure for our result. This however can evidently only be done when the development, either from the commencement or after a certain number of terms, becomes a *converging* series; that is, a series such that the more terms of it we add together the more nearly do we approach to some fixed and finite number, the equivalent of the entire series; so that by including more and more terms our sum may be made to differ from this finite number by as small a fraction as we please; and that the series may be of this kind, an essential condition is that its successive terms continually diminish, and thus tend towards zero.* The cube root of 9, determined in example 6 to thirteen places of decimals, is known to be true to that extent, because a little examination enables us to foresee that the leading term in the supplementary series rejected would be too small to have any significant figure in even the fourteenth place of decimals, and that moreover the series converges so rapidly, that, take as

* On the convergency of series, see the Essay on Logarithms, second Edition.

many terms as we may, the entire aggregate will never be so great as a unit in the thirteenth decimal place.

When however the series *diverges*, its successive terms receding more and more from zero, we shall have indication sufficient that no numerical approximation can be made by it to the value of the undeveloped expression; the supplementary correction, let us stop wherever we may, will in that case be the most important part of our result.

This supplementary correction however, being $(a + x)^m - s$, which can never, as in a converging series, be reduced to insignificance and rejected, renders the development totally useless for the purposes of numerical approximation; since its introduction merely presents us with the identity $(a + x)^m = (a + x)^m$, proving that our supposed development, when properly corrected, will turn out to be no development at all.

These observations may serve to reconcile what may at first appear to the inquiring student to be a remarkable discrepancy. For instance, suppose x in example 3 to be greater than b^2, then in the final equation the first member would, we know, be an imaginary quantity; but in the second member there is no trace of any imaginary quantity; on the contrary, the entrance of such a quantity is forbidden by the law of the series. In such an apparently anomalous case, it is of importance to know that the established truths of algebra are not really violated, that *both* members of the equation are equally imaginary, but that the indication of this is concealed under the "&c.," the second member being in the case supposed a diverging series.

It is easy to show that such a series will always be furnished by the development of $(a - x)^{\frac{m}{n}}$, whenever x exceeds a. For in this case the powers of $\frac{x}{a}$ or Q (p. 198) continually increase; and it is obvious that the numerical quantities

$$\frac{m}{n}, \frac{m-n}{2n}, \frac{m-2n}{3n}, \ldots\ldots \frac{m-kn}{(k+1)n}$$

must at length converge towards -1; for k must eventually become so large, in comparison with the fraction $\frac{m}{n}$, as to render

$$\frac{m - kn}{(k+1)n} \text{ or } \frac{\frac{m}{n} - k}{k+1}$$

as little different from $\frac{-k}{k+1}$ as we please: and this last expression, by the same increase of k, must come at length to differ from $\frac{-k}{k}$ by as little as we please; so that the numerical quantities adverted to really converge towards -1, after a certain stage, approaching to it indefinitely close. And as Q is > 1, it follows that the terms A, B, C, &c. (p. 198) must, after a certain stage, necessarily *increase.*

It may be observed finally, that it is not only in applications of the binomial theorem that what has the appearance of a development may, for particular values of the symbols, turn out when properly corrected to be no development at all. The common rule for extracting roots will furnish an abundance of such instances. Thus, in example 5, p. 46, the development, if properly corrected, would disappear when x is negative, and numerically greater than 1; for the series being then divergent, the correction could never be reduced to insignificance; it would necessarily consist of the original imaginary quantity, and of the real part already developed taken with a contrary sign.

(130.) We promised, at page 194, to present the student with another method of investigating the law of the coefficients in the Binomial Theorem. The method which we had in view is that which follows.

It has already been shown (page 185,) that the first term in the development of $(a + x)^{\frac{m}{n}}$ must always be $a^{\frac{m}{n}}$, and that the development involves none but integral powers of x; we may assume, therefore,

$$(a + x)^{\frac{m}{n}} = a^{\frac{m}{n}} + Bx + Cx^2 + Dx^3 + \&c.$$

or, changing x into y,

$$(a+y)^{\frac{m}{n}} = a^{\frac{m}{n}} + \mathrm{B}y + \mathrm{C}y^2 + \mathrm{D}y^3 + \&c.$$

hence, by subtraction,

$$(a+x)^{\frac{m}{n}} - (a+y)^{\frac{m}{n}} = \mathrm{B}(x-y) + \mathrm{C}(x^2-y^2) + \mathrm{D}(x^3-y^3) + \&c.$$

and consequently,

$$\frac{(a+x)^{\frac{m}{n}} - (a+y)^{\frac{m}{n}}}{(a+x)-(a+y)} = \frac{(a+x)^{\frac{m}{n}} - (a+y)^{\frac{m}{n}}}{x-y} = \mathrm{B} + \mathrm{C}(x+y) + \mathrm{D}(x^2+yx+y^2) + \mathrm{E}(x^3+yx^2+y^2x+y^3) + \&c.$$

If now we were to suppose $x=y$, the second member of this equation would present itself in a definite and intelligible form; but the first would become $\frac{0}{0}$, a fraction in which both numerator and denominator have vanished. As, however, this vanishing fraction has a definite value, shown by the second side of the equation, there can be no doubt that its ambiguous form, $\frac{0}{0}$, must have arisen from some common factor in the numerator and denominator of the original fraction having become 0, by putting in that fraction $x=y$.* If, then, we could discover this common factor, we should be able, by expunging it from both numerator and denominator, to free the fraction from all ambiguity; and the result of our hypothesis, $x=y$, would then be definite in *form* as well as in value. Now we shall be able to effect this by transforming our fraction into another of equivalent value, by means of the following substitutions.

Put $u=(a+x)^{\frac{1}{n}}$, $v=(a+y)^{\frac{1}{n}}$, $\therefore x-y=u^n-v^n$, and consequently,

$$\frac{u^m-v^m}{u^n-v^n} = \mathrm{B} + \mathrm{C}(x+y) + \mathrm{D}(x^2+yx+y^2) + \mathrm{E}(x^3+yx^2+y^2x+y^3) + \&c.$$

Now both numerator and denominator of the first member of

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* For the theory of *vanishing fractions*, see the volume on the *Theory of Equations*, second edition, page 113.

this equation are divisible by $u - v$; and $u - v$ is the very factor which vanishes for $x = y$, as is at once seen by referring to the substitutions just proposed. This factor will be removed by actually dividing numerator and denominator by $u - v$, which reduces the fraction to

$$\frac{u_m{}^{-1} + vu^{m-2} + v^2u^{m-3} + \ldots\ldots v^{m-1}}{u^{n-1} + vu^{n-2} + v^2u^{n-3} + \ldots\ldots v^{n-1}}* = \mathrm{B} + \mathrm{C}\,(x+y) + \mathrm{D}\,(x^2 + yx + y^2) + \&c.$$

Introducing now the proposed hypothesis, $x = y$, which leads to $v = u$, we have

$$\frac{mv^{m-1}}{nv^{n-1}} = \frac{mv^m}{nv^n} = \mathrm{B} + 2\,\mathrm{C}x + 3\,\mathrm{D}x^2 + 4\,\mathrm{E}x^3 + \&c.;$$

that is, by restoring the value of v,

$$\frac{m}{n} \cdot \frac{(a+x)^{\frac{m}{n}}}{a+x} = \mathrm{B} + 2\,\mathrm{C}x + 3\,\mathrm{D}x^2 + 4\,\mathrm{E}x^3 + \&c.$$

Multiply both members by $a + x$; and then, instead of $(a + x)^{\frac{m}{n}}$ in

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* This may be proved as follows: From the common expression for the sum of m terms of a geometrical series, we have

$$\frac{r^m - 1}{r - 1} = r^{m-1} + r^{m-2} + r^{m-3} \ldots\ldots + r + 1.$$

Let $r = \frac{u}{v}$, then

$$\frac{\frac{u^m}{v^m} - 1}{\frac{u}{v} - 1} = \left(\frac{u}{v}\right)^{m-1} + \left(\frac{u}{v}\right)^{m-2} + \left(\frac{u}{v}\right)^{m-3} \ldots\ldots + \frac{u}{v} + 1.$$

The first member, by multiplying numerator and denominator by v, is $\dfrac{\frac{u^m}{v^{m-1}} - v}{u - v}$: hence, multiplying both members by v^{m-1}, we have

$$\frac{u^m - v^m}{u - v} = u^{m-1} + vu^{m-2} + v^2u^{m-3} + \ldots + v^{m-2}u + v^{m-1}$$

as in the text.

the first member, write its developed form with which we set out, and we shall have

$$\frac{m}{n}a^{\frac{m}{n}} + \frac{m}{n}Bx + \frac{m}{n}Cx^2 + \frac{m}{n}Dx^3 + \&c. =$$

$$\left.\begin{matrix}Ba + 2Ca \\ B\end{matrix}\right| x + \left.\begin{matrix}3\,Da \\ 2\,C\end{matrix}\right| x^2 + \left.\begin{matrix}4\,Ea \\ 3\,D\end{matrix}\right| x^3 + \&c.$$

Hence, by the theorem at page 182.

$$Ba = \frac{m}{n}a^{\frac{m}{n}}, \text{ therefore } B = \frac{m}{n}a^{\frac{m}{n}-1}$$

$$2\,Ca + B = \frac{m}{n}B \;\ldots\ldots\ldots\; C = \frac{(\frac{m}{n}-1)\,B}{2a}$$

$$3\,Da + 2\,C = \frac{m}{n}C, \text{ therefore } D = \frac{(\frac{m}{n}-2)\,C}{3a}$$

$$4\,Ea + 3\,D = \frac{m}{n}D \;\ldots\ldots\ldots\ldots\; E = \frac{(\frac{m}{n}-3)\,D}{4a}$$

&c. &c.

Consequently,

$$(a+x)^{\frac{m}{n}} = a^{\frac{m}{n}} + \frac{m}{n}a^{\frac{m}{n}-1}x + \frac{\frac{m}{n}(\frac{m}{n}-1)}{2}a^{\frac{m}{n}-2}x^2 +$$

$$\frac{\frac{m}{n}(\frac{m}{n}-1)(\frac{m}{n}-2)}{2\,.\,3}a^{\frac{m}{n}-3}x^3 + \&c.$$

In this demonstration m and n may obviously be any whole numbers whatever, and m may be either positive or negative.

(131.) With the aid of the binomial theorem, we may, by a simple and elegant process, obtain the development of a^x in a series according to the ascending powers of x. The quantity a^x is called an *exponential* quantity, and the development of which we speak is called the *exponential theorem:* this theorem we shall now investigate, on account of its importance in the theory of logarithms, and in other departments of analysis.

THE EXPONENTIAL THEOREM.

We are here required to exhibit the development of a^x according to the ascending powers of x. We shall commence by showing that the proposed form of development is possible.

Put $a = 1 + b$, $\therefore a^x = (1+b)^x$, and, by the binomial theorem,

$$(1+b)^x = 1 + xb + \frac{x(x-1)}{2}b^2 + \frac{x(x-1)(x-2)}{2.3}b^3$$

$$+ \frac{x(x-1)(x-2)(x-3)}{2.3.4}b^4 + \text{\&c.}$$

and it is obvious, that if the multiplications indicated by the numerators in the right-hand member of this equation were actually executed, the result would be a series of monomials in x, in x^2, in x^3, &c., which we might arrange in a regular ascending order. The term in x is ascertainable at once from mere inspection, it is

$$\{b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \text{\&c.}\}\, x;$$

so that we may safely conclude that a^x may be developed in the form

$$a^x = 1 + \{b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \text{\&c.}\}\, x + Bx^2 + Cx^3 + \text{\&c.}$$

Having thus seen the possibility of the proposed development, let us assume

$$\left.\begin{aligned} a^x &= 1 + Ax + Bx^2 + Cx^3 + \text{\&c.} \\ \text{in like manner, } a^y &= 1 + Ay + By^2 + Cy^3 + \text{\&c.} \end{aligned}\right\} \ldots\ldots\ldots [1];$$

also $a^{x+y} = 1 + A(x+y) + B(x+y)^2 + C(x+y)^3 + \text{\&c.} \ldots [2]$.

By the binomial theorem, the coefficient of y in [2] is

$$A + 2Bx + 3Cx^2 + 4Dx^3 + \text{\&c.} \;\ldots\ldots\ldots [3];$$

and the coefficient of y in $a^x + a^y$, as obtained from multiplying the series for a^x in [1] by the coefficient A in the series for a^y, is

$$A + A^2x + ABx^2 + ACx^3 + \text{\&c.} \;\ldots\ldots\ldots [4].$$

Hence, since a^{x+y} and $a^x + a^y$ are identical, these coefficients of y are identical (125); and therefore (125)

$$2B = A^2 \therefore B = \frac{A^2}{2}$$

$$3C = AB \therefore C = \frac{A^3}{2.3}$$

$$4D = AC \therefore D = \frac{A^4}{2.3.4}$$

&c. &c.

Consequently [1]

$$a^x = 1 + Ax + \frac{A^2x^2}{2} + \frac{A^3x^3}{2.3} + \frac{A^4x^4}{2.3.4} + \&c.,$$

which is the *exponential theorem*, and in which

$$A = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.*$$

* The foregoing development is obviously infinite, whatever be the value of x. The development of a binomial, when arranged according to the powers of one of its terms, as we have already seen, is always finite when the exponent is a positive integer; the above theorem shows that this will never be the case when the arrangement is according to the powers of the exponent.

CHAPTER VII.

ON LOGARITHMS.

(132.) LOGARITHMS are certain numbers invented by Lord Napier for the purpose of facilitating arithmetical computations, by reducing the numerical processes of multiplication and division to the more simple operations of addition and subtraction. To understand how this is effected, we must consider every positive number as a power, whole or fractional, of some assumed root fixed upon at pleasure: from this root all positive numbers are supposed to be generated, by involution or evolution; and it is the *exponent* of this root which is called the *logarithm* of the number or power generated. A table therefore containing the logarithms of the numbers 1, 2, 3, 4, &c. is nothing more than a table of the several exponents which the assumed root must take to produce the numbers 1, 2, 3, 4, &c. Thus, if a be any assumed number, and such values be successively given to x that will make $a^x = b$, $a^x = c$, $a^x = d$, &c., then these different values of x are the logarithms of b, c, d, &c. respectively: If $x = 0$, then $a^x = 1$, whatever the value of a, (Art. 39, Chap. I); hence the logarithm of 1 is always 0.

(133.) The assumed root a is called the *base* of the system of logarithms, and from different bases different systems of logarithms must evidently arise; but it has been found to be most convenient to assume 10 for the base, and upon this assumption all our modern logarithmic tables are constructed. The advantage of the base 10 over every other base will be seen hereafter.

(134.) Assuming therefore $a = 10$, we have

$$10^0 = 1,\ 10^1 = 10,\ 10^2 = 100,\ 10^3 = 1000, \text{\&c.}$$

that is, the log of 1 is 0, the log of 10 is 1, the log of 100 is 2, the log of 1000 is 3, &c.

Also $10^{-1} = \frac{1}{10}$, $10^{-2} = \frac{1}{100}$, $10^{-3} = \frac{1}{1000}$, &c.;
that is,
the log of $\frac{1}{10}$ is -1, the log of $\frac{1}{100}$ is -2, the log of $\frac{1}{1000}$ is -3, &c.

(135.) Hence, since the log of 1 is 0, and the log of 10 is 1, it follows that the log of any number between 0 and 10 must lie between 0 and 1; and in the same manner the log of any number between 10 and 100 must lie between 1 and 2, &c., and therefore these logarithms may be either accurately found, or may be approximated to, to any degree of precision. But, before we explain the method of obtaining this approximate value of the logarithm of any given number, it will be convenient to establish the following characteristic properties of logarithms.

(136.) THEOREM 1. The sum of the logarithms of any two numbers is equal to the logarithm of their product.

Let b be any number, and let its logarithm be x; and let c be any other number, whose logarithm is x'; then $a^x = b$, and $a^{x'} = c$; and by multiplying, $a^{x+x'} = bc$; that is, $x + x'$ is the logarithm of bc.

Cor. 1. Hence the sum of the logarithms of any number of numbers is the logarithm of their product.

Cor. 2. Therefore n times the logarithm of any number is the logarithm of its nth power.

From the preceding theorem and its corollaries, it is easy to foresee the use that may be made of a table of logarithms, or *exponents*, in arithmetical multiplication or involution: If we take from such a table the logarithms of the component factors—these logarithms being written in the table against the factors themselves—and then add them together, the result will be the logarithm of the product, and will therefore be found in the table opposite to that product, which thus becomes known.

THEOREM 2. The difference of the logarithms of any two numbers is equal to the logarithm of their quotient.

For, since $a^x = b$, and $a^{x'} = c$, by dividing,

$$\frac{a^x}{a^{x'}} = a^{x-x'} = \frac{b}{c}; \text{ that is, } x - x' = \log \frac{b}{c}.$$

It thus appears that, by the help of a table of logarithms, arithmetical division may be replaced by simple subtraction; we shall only have to take out of the table the logarithm of the dividend, and the logarithm of the divisor, to subtract the latter from the former, and then to take from the table the number whose logarithm is the *remainder;* this number will be the quotient sought.

THEOREM 3. The nth part of the logarithm of any number is equal to the logarithm of its nth root.

For if $a^x = b$, $a^{\frac{x}{n}} = b^{\frac{1}{n}}$, that is, $\frac{x}{n} = \log b^{\frac{1}{n}}$.

Hence, to find the nth root of a number, it will be only necessary to take the logarithm of that number from a table, to divide this log by n, and then to take from the table the number which has the quotient for its logarithm.

THEOREM 4. If there be any series of quantities in geometrical progression, their logarithms will be in arithmetical progression.

Let the geometrical progression be b, nb, n^2b, n^3b, &c., and let x be the log of b, and z the log of n; then $a^x = b$, and $a^z = n$, therefore the progression is the same as

$$a^x, a^{x+z}, a^{x+2z}, a^{x+3z}, \text{\&c.},$$

where the logarithms x, $x + z$, $x + 2z$, $x + 3z$, &c. are in arithmetical progression.

PROBLEM.

(137.) To find the logarithm of any given number.

Let N be any given number, then it is required to find the value of x in terms of a and N, so that we may have $a^x = \text{N}$. For this purpose, put $a = 1 + m$, and $\text{N} = 1 + n$; then $(1 + m)^x = 1 + n$, and therefore $(1 + m)^{xy} = (1 + n)^y$, whatever be the value of y; hence, by developing,

$$1 + xym + \frac{xy(xy-1)}{2}m^2 + \frac{xy(xy-1)(xy-2)}{2.3}m^3 + \&c. =$$

$$1 + yn + \frac{y(y-1)}{2}n^2 + \frac{y(y-1)(y-2)}{2.3}n^3 + \&c.;$$

or expunging the 1, and dividing by y, we have

$$x\,(m + \frac{xy-1}{2}m^2 + \frac{(xy-1)(xy-2)}{2.3}m^3 + \&c.) =$$

$$n + \frac{y-1}{2}n^2 + \frac{(y-1)(y-2)}{2.3}n^3 + \&c.$$

Suppose now $y = 0$, or which is the same thing, equate the terms independent of y (125), and we have

$$x\,(m - \frac{m^2}{2} + \frac{m^3}{3} - \&c.) =$$

$$n - \frac{n^2}{2} + \frac{n^3}{3} - \&c.$$

whence $x = \log(1+n) = \dfrac{n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \&c.}{m - \frac{1}{2}m^2 + \frac{1}{3}m^3 - \&c.}$;

or substituting for n and m their respective values $\text{N} - 1$, and $a - 1$, we have

$$\log \text{N} = \frac{(\text{N}-1) - \frac{1}{2}(\text{N}-1)^2 + \frac{1}{3}(\text{N}-1)^3 - \&c.}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.}$$

We have deduced this result without any aid from the exponential theorem. But by introducing that theorem we shall arrive at the conclusion above much more speedily. Thus, raising each side of the proposed equation to the power y, we shall have

$$a^{xy} = \text{N}^y,$$

that is, by the exponential theorem,

$$1 + \text{A}xy + \text{A}^2\frac{x^2y^2}{2} + \&c. = 1 + \text{A}'y + \text{A}'^2\frac{y^2}{2} + \&c.$$

hence, equating the coefficients of y,

$$\text{A}x = \text{A}', \therefore x = \frac{\text{A}'}{\text{A}} = \frac{(\text{N}-1) - \frac{1}{2}(\text{N}-1)^2 + \frac{1}{3}(\text{N}-1)^3 - \&c.}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.}$$

Hence we have, as before, the value of log N in terms of N and a. This expression for the logarithm of any number is however of but little use in constructing a table of logarithms, on account

of the slow convergency of the terms of the numerator; we must therefore investigate a method of converting it into other expressions that may be more suitable for this purpose.

(138.) Since the value of the denominator of the above fraction depends entirely upon the value of the base a, it will accordingly differ in different systems of logarithms; but the numerator, being wholly independent of the base a, must be the same in every system.

(139.) The reciprocal of the denominator, that is to say the fraction $\frac{1}{A}$, is called the *modulus* of the system, and is usually represented by M; so that we have

$$\log(1+n) = M(n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 + \&c.);$$

and supposing n negative,

$$\log(1-n) = M(-n - \tfrac{1}{2}n^2 - \tfrac{1}{3}n^3 - \tfrac{1}{4}n^4 - \&c.);$$

hence, subtracting this equation from the former,

$$\log(1+n) - \log(1-n) = \log\frac{1+n}{1-n}\ (\text{theo. } 2) =$$

$$2M(n + \tfrac{1}{3}n^3 + \tfrac{1}{5}n^5 + \&c.)$$

The series here deduced is obviously more convergent than that originally obtained, n being in both cases supposed to be less than unity. But since for the actual construction of a table, we must be supplied with series that will be rapidly convergent for all values of the proposed number, from 1 upwards, some algebraic artifice must be employed to convert the preceding expression into the desired form. Now this consists in substituting $\frac{P+1}{P}$ for $\frac{1+n}{1-n}$, or in equating these two fractions together; from which equation we get

$$n = \frac{1}{2P+1}$$

and consequently, by making these substitutions in the result obtained above, we have

$$\log(P+1)-\log P=2M\left(\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\text{\&c.}\right);$$

$$\therefore \log(P+1)=2M\left(\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\text{\&c.}\right)+\log P.$$

Hence, if log P be given, the log of the next greater number may be found by this series, which converges* very fast, and therefore, since the log 1 is given $=0$, we can from this get log 2, and thence the logs of all the natural numbers in succession.

(140.) *To construct a Table of Napierian, or Hyperbolic Logarithms.*

But before we can employ the series which we have just given for the purpose of forming a table, we must assign some value to M, and as this value may be arbitrary, since the base a on which it entirely depends is arbitrary, let it be 1; which is the value assumed by Napier, the inventor of logarithms; we shall then have

$$\log(P+1)=2\left(\frac{1}{2P+1}+\frac{1}{3(2P+1)^3}+\frac{1}{5(2P+1)^5}+\text{\&c.}\right)+\log P,$$

and making P successively equal to 1, 2, 3, &c. we shall have

---

* As already remarked at p. 204, a series is said to converge when its value is finite, and its terms diminish in such a way that the more of them we take, setting out from the first, the nearer will their sum approach to that of the entire series. The more rapid the rate of diminution is, the greater is the convergency of the series, that is, the less will any proposed number of the leading terms differ from the whole sum. (See the new edition of the 'Essay on Logarithms.')

$\log\ 2 = 2\,(\frac{1}{3} + \frac{1}{3^4} + \frac{1}{5\,.\,3^5} + \&c.)\ .\ .\ .\ .\ .\ = \cdot 6931472$

$\log\ 3 = 2\,(\frac{1}{5} + \frac{1}{3\,.\,5^3} + \frac{1}{5^6} + \&c.) + \log 2\ .\ .\ = 1\cdot 0986123$

$\log\ 4 = 2 \log 2$ (theo. 1) . . . . $= 1\cdot 3862944$

$\log\ 5 = 2\,(\frac{1}{9} + \frac{1}{3\,.\,9^3} + \frac{1}{5\,.\,9^5} + \&c.) + \log 4\ .\ .\ = 1\cdot 6094379$

$\log\ 6 = \log 2 + \log 3$ . . . . $= 1\cdot 7917595$

$\log\ 7 = 2\,(\frac{1}{13} + \frac{1}{3\,(13)^3} + \frac{1}{5\,(13)^5} + \&c.) + \log 6 = 1\cdot 9459101$

$\log\ 8 = 3 \log 2$ . . . . $= 2\cdot 0794415$

$\log\ 9 = 2 \log 3$ . . . . $= 2\cdot 1972246$

$\log 10 = \log 2 + \log 5$ . . . . $= 2\cdot 3025851$

&c. &c.

By proceeding in this way, the logarithms of all the natural numbers according to this particular system may be obtained, and that too without knowing the value of the base, a, of which these logarithms are the several exponents, since this base enters into the general expression for log N only to form the modulus M. In the Napierian system the base is of that particular value which satisfies the condition $M = 1$; but tables constructed conformably to this system, in which we see the logarithm of 10 is 2·3025851, are by no means so advantageous for the general purposes of computation as those in which the logarithm of 10 is 1, as has been before observed; we shall therefore show how

(141.) *To construct a Table of Common Logarithms.*

In the system of common logarithms, the value of M is to be determined from the supposition that the base a is 10; and, as the value of the logarithms in any system depends entirely on the value of 2M, if this value in one system be r times that in another, the logarithm of any number by the former system must be r times that by the latter, and *vice versâ;* now, in the hyper-

bolic system, the logarithm of 10 is 2·3025851, therefore, in order that the logarithm of 10 may be 1, the value of 2M, in the common system, must be the 2·3025851th part of its value in the hyperbolic system; but in this system $2M = 2$, therefore, in the common system, $2M = \frac{2}{2\cdot 3025851} = \cdot 86858896$.* Hence, to construct a table of common logarithms, we have

$$\log(P+1) = \cdot 86858896\left(\frac{1}{2P+1} + \frac{1}{3(2P+1)^3} + \frac{1}{5(2P+1)^5} + \&c.\right) + \log P;$$

that is, by making $P = 1, 2, 3$, &c. successively,

$$\log 2 = \cdot 86858896\left(\tfrac{1}{3} + \frac{1}{3^4} + \frac{1}{5.3^5} + \&c.\right) \ldots = \cdot 3010300$$

$$\log 3 = \cdot 86858896\left(\tfrac{1}{5} + \frac{1}{3.5^3} + \frac{1}{5^6} + \&c.\right) + \log 2 = \cdot 4771213$$

$$\log 4 = 2\log 2 \ldots\ldots = \cdot 6020600$$

$$\log 5 = 8\cdot 6858896\left(\tfrac{1}{9} + \frac{1}{3.9^3} + \frac{1}{5.9^5} + \&c.\right) + \log 4 = \cdot 6989700$$

$$\log 6 = \log 2 + \log 3 \ldots\ldots = \cdot 7781513$$

$$\log 7 = \cdot 86858896\left(\tfrac{1}{13} + \frac{1}{3(13)^3} + \frac{1}{5(13)^5} + \&c.\right) + \log 6 = \cdot 8450980$$

$$\log 8 = 3\log 2 \ldots\ldots = \cdot 9030900$$

$$\log 9 = 2\log 3 \ldots\ldots = \cdot 9542426$$

$$\log 10 = \log 2 + \log 5 \ldots\ldots = 1\cdot 0000000$$

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* As the logarithms of the same number in two different systems are to each other as the constant moduli of those systems, it follows that, if of any number N the logarithms in two systems whose bases are a and a' be represented by log N and log 'N, we shall have

$$\frac{\log N}{\log' N} = \frac{\log a}{\log' a} = \frac{1}{\log' a};$$

$$\therefore \log N = \frac{\log' N}{\log' a},$$

and thus we have only to divide the logarithm of any number by the logarithm of a in order to obtain the logarithm of the same number in

and in this manner may a table of common logarithms be constructed; and since the logarithms in the hyperbolic system are 2·3025851 times those in the common system, from having a table of the one we may form from it a table of the other.

(142.) In common logarithmic tables, the decimals only of the logarithms are inserted, and the integral part, which is called the *index* or *characteristic*, is omitted, because this integral part is always known from the number itself, whose logarithm is sought; for if this number consist of two integral figures, it must be either 10, or some number between 10 and 100, and, consequently, its logarithm must be either 1, or some number between 1 and 2, that is, the integral part must be 1. In the same manner, if the number consist of three integers, the integral part of its logarithm must be 2, &c.; so that the index or characteristic is always equal to the number of integral figures in the proposed number, *minus* 1.

(143.) It also follows, that in this system the logarithm of any number, and that of one 10 times as great, differ only in the index, the decimal part being the same; so that the decimal parts of the logarithms of all numbers consisting of the same figures remain the same, whether those figures are integers or decimals, or partly integral and partly decimal; thus:

$$\log 3526 \quad \ldots\ldots\ldots\ldots \quad = 3{\cdot}5472823$$

$$\log 352{\cdot}6 = \log \frac{3526}{10} = \log 3526 - 1 = 2{\cdot}5472823$$

$$\log 35{\cdot}26 = \log \frac{352{\cdot}6}{10} = \log 352{\cdot}6 - 1 = 1{\cdot}5472823$$

$$\log 3{\cdot}526 = \log \frac{35{\cdot}26}{10} = \log 35{\cdot}26 - 1 = {\cdot}5472823$$

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the system whose base is a; so that to pass from Napierian to common logarithms, we have

$$\text{com. log } N = \frac{\text{nap. log } N}{\text{nap.log } 10} = \frac{\text{nap. log } N}{2{\cdot}3025851}.$$

$$\log \cdot 3526 = \log \frac{3\cdot 526}{10} = \log 3\cdot 526 - 1 = \bar{1}\cdot 5472823$$

$$\log \cdot 03526 = \log \frac{\cdot 3526}{10} = \log \cdot 3526 - 1 = \bar{2}\cdot 5472823$$

$$\log \cdot 003526 = \log \frac{\cdot 03526}{10} = \log \cdot 03526 - 1 = \bar{3}\cdot 5472823$$

&c. &c.

It thus appears that the integral part of the logarithm of any number may always be found by counting the number of places that the leading significant figure is removed from the place of units; if this leading figure lie to the left of the units' place, the integer thus determined will be *plus*; if it lie to the right, it will be *minus*; and if it occupy the units' place itself, it will be 0.

It is plain that the logarithm of the reciprocal of any number is the logarithm of the number itself, with the minus sign prefixed: thus,

$$\log \frac{1}{3526} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; = -3\cdot 5472823$$

$$\log \frac{1}{352\cdot 6} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; = -2\cdot 5472823$$

$$\log \frac{1}{35\cdot 26} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; = -1\cdot 5472823$$

&c. &c.

Also,

$$\log \frac{1}{\cdot 03526} \;.\;.\;.\; = -\bar{2}\cdot 5472823 = 2 - \cdot 5472823$$

&c. &c. &c.

We may now perceive the superiority of this system over every other, since the above property, which belongs only to this particular system, will evidently greatly facilitate the construction of a table, it being only necessary to compute the logarithms of the *whole numbers*; whereas, in every other system, each particular number, whether integral or decimal, requires a particular logarithm.

These advantages of the present system were suggested to Lord Napier by Mr. Briggs, soon after the invention of logarithms: and on this account the improved logarithms, of which all the tables in common use consist, are sometimes called Briggs's logarithms.

Napier had, however, already contemplated a like change in his own system; so that, although Briggs performed the calculations of the new system, and indeed first publicly announced its superior advantages, yet Napier appears to have entertained a clear conception of these advantages previously to Briggs's communication to him.*

To determine the Napierian Base.

(144.) We have already remarked that, in Napier's system, the base was that particular value of a which satisfied the condition

$$(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\&c.=1.$$

Let us call this particular value e, then, by the exponential theorem, (p. 210),

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2\,.\,3}+\frac{x^4}{2\,.\,3\,.\,4}+\&c.$$

which for $x=1$ gives for the base e the value

$$e=1+1+\frac{1}{2}+\frac{1}{2\,.\,3}+\frac{1}{2\,.\,3\,.\,4}+\&c.$$

* This fact, however, cannot lessen the great merit of Briggs, nor dispossess him of his claim to the title of "the great improver of logarithms." In the recently published 'Life of Lord Napier,' this question of the *improvement* of logarithms is no doubt amply discussed. But the author of this treatise has not had an opportunity of consulting that costly work.

which may be thus calculated:

$$\begin{array}{rrl}
2 & \ldots & = a \\
\tfrac{1}{2} = & \cdot 5 \ldots & = b \\
\tfrac{1}{3} b = & \cdot 1666666666 & = c \\
\tfrac{1}{4} c = & 416666666 & = d \\
\tfrac{1}{5} d = & 83333333 & = e \\
\tfrac{1}{6} e = & 13888888 & = f \\
\tfrac{1}{7} f = & 1984127 & = g \\
\tfrac{1}{8} g = & 248016 & = h \\
\tfrac{1}{9} h = & 27557 & = i \\
\tfrac{1}{10} i = & 2755 & = k \\
\tfrac{1}{11} k = & 250 & = l \\
\tfrac{1}{12} l = & 21 & = m \\
\hline
 & 2 \cdot 718281828 &
\end{array}$$

hence the value of the Napierian or hyperbolic base is 2·718281828.*

It is worthy of notice, that if in the development of a^x (p. 211), we put $\frac{1}{A}$ for x, the result will be identical with the foregoing series for e; therefore, *whatever be a*, we have

$$a^{\frac{1}{A}} = e, \therefore \frac{1}{A} \log a = \log e,$$

and, taking Napierian logarithms,

$$\frac{1}{A} = M = \frac{1}{\text{Nap. log } a},$$

as determined from other considerations at page 219.

* If in the preceding development of e^x we put $x = -1$, we shall have

$$e^{-1} = \frac{1}{e} = 1 - 1 + \tfrac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} - \text{\&c.}$$

so that this series is the reciprocal of the former.

What is here said upon the subject of logarithms is doubtless sufficient to convey to the student a correct notion of their nature and principal properties, as also of the practicability of constructing a table of them to any extent. But the labour of actually computing a whole table of logarithms by means of the series here investigated would be great in the extreme; they are, however, susceptible of a variety of transformations much better adapted to the use of the computer. To explain and exhibit these would carry us too far into the business of series, and would occupy too large a portion of this treatise. But the inquiring student, who is desirous of ample information upon the most expeditious methods of calculating a table of logarithms, may refer to the second edition of the author's 'Essay on the Computation of Logarithms;' and the manner of using a table thus constructed is fully explained in the introduction prefixed to the 'Mathematical Tables.'

APPLICATION OF LOGARITHMS.

LOGARITHMICAL ARITHMETIC.

(145.) From what has been already said on the nature and properties of logarithms, the following operations, performed by means of a table, will be readily understood without any further explanation.

EXAMPLE 1. Multiply 23·14 by 5·062.

Here the log of 23·14 in the tables* is

	1·3643634
log of 5·062	·7043221
	2·0686855 = log of 117·1347 = the product.

* The tables employed are Young's 'Mathematical Tables,' computed to seven places of decimals.

2. Divide ·06314 by ·007241.

Here the log of ·06314 is $\bar{2}·8003046$

log of ·007241 $\bar{3}·8597985$

·9405061 = log of 8·719792 = the quotient.

Required the fourth power of ·09163.

Here the log of ·09163 is $\bar{2}·9620377$

4

$\bar{5}·8481508$ = log of ·0000704938.

Required the tenth root of 2.

Here $\frac{\log 2}{10} = \frac{0·30103}{·10} = ·030103 = \log 1·07179.$

Required the value of $\frac{8^5 \times \sqrt[3]{7}}{\sqrt[5]{6}}$.

Here $5 \log 8 + \frac{1}{3} \log 7 - \frac{1}{5} \log 6 = 4·51545 + ·2816993 - ·1556302$ $= 4·6415191 = \log 43804·53.$

Required the value of $\frac{24^6 \times \sqrt[3]{17}}{4821 \times 6^4}$.

Ans. 78·64561.

Required the value of $\sqrt{\frac{284\sqrt[3]{621}}{43^3}}$.

Ans. $\frac{1}{5·72833}$.

The few examples here given are sufficient to show the great advantages of logarithms in abridging arithmetical labour, in which indeed consists their principal, although not their only value. There are many analytical researches which it would be impossible to carry on without their aid, and many others in which the introduction of logarithmic formulas greatly facilitates the deductive process. It would be easy to propose questions, the solutions of which might be comprised in a few lines by logarithms, but which, without their aid, would occupy many volumes of closely-printed figures. The following is a striking example:

10 §

Let there be a series of numbers commencing with 2, and such that each is the square of that which immediately precedes it: it is required to determine the number of figures which the 25th term would consist of.

The series proposed is obviously

$$2,\ 2^2,\ 2^4,\ 2^8,\ 2^{16},\ \&c.$$

the exponent of the nth term being 2^{n-1}, and therefore the exponent of the 25th term is $2^{24} = 16777216$; consequently, calling the 25th term x, we have

$$x = 2^{16777216},\ \text{whence } \log x = 16777216 \log 2$$
$$= 16777216 \times \cdot 30103$$
$$= 5050445 \cdot 33248;$$

hence, since the index or characteristic of this logarithm is 5050445, the number answering to it must consist of 5050446 figures; so that the number x, if printed, would fill nine volumes of 350 pages each, allowing 40 lines to a page and 40 figures to a line.

ON EXPONENTIAL EQUATIONS: RULE OF TRIAL AND ERROR.

(146.) An exponential equation is an equation in which the unknown term is expressed in the form of a power with an unknown index; thus, the following are exponential equations:

$$a^x = b,\ x^x = a,\ a^{b^x} = c,\ \&c.$$

(147.) When the exponential is of the form a^x, the value of x is readily found by logarithms; for if $a^x = b$, we have

$$x \log a = \log b,\ \therefore\ x = \frac{\log b}{\log a}$$

Also, if $a^{b^x} = c$, put $b^x = y$; then $a^y = c$, whence $y \log a = \log c$;

$$\therefore\ y = \frac{\log c}{\log a} : \text{put this} = d, \text{ then } b^x = d,\ \therefore\ x = \frac{\log d}{\log b}.$$

(148.) But if the equation be of the form $x^x = a$, then the value of x may be obtained by the following rule of *double position.*

RULE. Find by trial two numbers as near the true value of x as you can, and substitute them separately for x; then, as the difference of the results is to the difference of the two assumed numbers, so is the difference of the true result, and either of the former, to the difference of the true number and the supposed one belonging to the result last used; this difference therefore being added to the supposed number or subtracted from it, according as it is too little or too great, will give the true value nearly.

And if this near value be substituted for x, as also the nearest of the first assumed numbers, unless a number still nearer be found, and the above operations be repeated, we shall obtain a still nearer value of x; and in this way we may continually approximate to the true value of x.

The preceding rule, which is applicable to a great variety of inquiries, will lead to the true value sought—and not to an approximation merely—whenever the unknown quantity is involved in an expression of the first degree only, that is, when its value depends on a simple equation: but when the higher powers of the unknown are involved, or when it enters as an exponent, we can do no more than approximate to its value by this method:

Let x be the number sought, and let it be determinable by a condition of the first degree, as $ax + b = c$. Let x', x'', be two trial values of x, the corresponding results being c', c'': then we have the three equations

$$ax + b = c,\ ax' + b = c',\ ax'' + b = c''$$

from which we immediately deduce the three,

$$a(x - x') = c - c',\ a(x - x'') = c - c'',\ a(x' - x'') = c' - c''$$

and thence for a, the three expressions

$$\frac{c' - c''}{x' - x''} = \frac{c - c'}{x - x'} = \frac{c - c''}{x - x''},$$

which furnish the algebraic statement of the rule: and are evidently

true whether the trial numbers x', x'', are near the truth or not. But when the proposed equation involves the higher powers of x, then the errors $x - x'$ and $x - x''$ must occur in the higher powers; so that the rule, by supposing these higher powers absent, either falls short of the truth or exceeds it, by a quantity however which becomes more and more unimportant as the original errors become diminished.

EXAMPLES.

1. Given $x^x = 100$, to find an approximate value of x.

$$\text{Here } x \log x = \log 100 = 2,$$

and upon trial x is found to lie between 3 and 4;

$\therefore$ substituting each of these, we have

$$3 \log 3 = 1{\cdot}4313639$$
$$\text{and } 4 \log 4 = 2{\cdot}4082400$$

$$\therefore \cdot 9768761 = \text{difference of results.}$$

$$\therefore \cdot 9768761 : 1 :: \cdot 4082400 : \cdot 418,$$

whence $4 - \cdot 418 = 3{\cdot}582 = x$ nearly.

Now this value is found, upon trial, to be rather too small; and 3·6 is found to be rather too great; therefore, substituting each of these, we have

$$3{\cdot}582 \log 3{\cdot}582 = 1{\cdot}9848779$$
$$3{\cdot}6 \log 3{\cdot}6 \quad = 2{\cdot}0026890$$

$$\therefore \cdot 0178111 = \text{diff. of results.}$$

$$\therefore \cdot 0178111 : \cdot 018 :: \cdot 002689 : \cdot 002717,$$

whence $3{\cdot}6 - \cdot 002717 = 3{\cdot}597283 = x$ very nearly.

The operation of solving the equation $x^x = a$ may be conducted differently, by using logarithms throughout; thus, in the equation $x \log x = \log a$, call $\log x$, x'; and $\log a$, a'; then $xx' = a'$, $\therefore \log x + \log x' = \log a'$, that is, $x' + \log x' = \log a'$; hence we

have to find a number x', which, when increased by its log, shall be equal to log a', which may be effected by the rule of position before given.

Thus, taking the same example as before, viz. $x^x = 100$, we have $\log 100 = 2 = a'$, and $\log 2 = \cdot 3010300$; $\therefore x' + \log x' = 3010300$, and the nearest value of x' in the tables below the true value is $\cdot 55597$, which added to its log $\bar{1}\cdot 7450514$, gives $\cdot 3010214$, and $\therefore$ the nearest value above the truth is $\cdot 55598$, which, added to its log $\bar{1}\cdot 740592$, gives $\cdot 3010392$; hence, by the rule,

$$\begin{array}{rl} 3010392 & \quad 3010300 \\ 3010214 & \quad 3010214 \\ \hline 178 = \text{diff. of results.} & \quad 86 \end{array}$$

$$\therefore 178 : 1 :: 86 : 483,$$

consequently $x' = \cdot 55597483 = \log x$, $\therefore x = 3\cdot 597284$.

If a be less than unity this solution fails, since a' is then negative, and therefore the $\log a'$ is unassignable. But if we put $x = \frac{1}{y}$, and $a = \frac{1}{b}$, we shall have, by substitution, the equation $b^y = y$, $\therefore y \log b = \log y$; put $\log b = b'$, and $\log y = y'$, then $yb' = y'$, $\therefore \log y + \log b' = \log y'$, or $y' + \log b' = \log y'$; whence y' may be found by the preceding rule.

2. Given $x^x = 5$, to find an approximate value of x.

Ans. $x = 2\cdot 1289$.

3. Given $x^x = 2000$, to find an approximate value of x.

Ans. $x = 4\cdot 8278$.

4. Given $x^x = 123456789$, to find an approximate value of x.

Ans. $x = 8\cdot 640027$.

SCHOLIUM.

The preceding rule of double position, or *trial and error* as it is sometimes called, may frequently be applied with advantage to the determination of approximate values for the unknown quantity in other complicated equations as well as in exponential

ones; as already intimated at page 227. An example or two will sufficiently illustrate the use and universality of the method.

1. Required an approximate value of x that will satisfy the condition $x^3 - 2x = 5$.

Here x is found upon trial to lie between 2 and 3; therefore, substituting these for x, successively, we have

$$8 - 4 = 4$$

$$27 - 6 = 21$$

$$\therefore 17 = \text{diff. of results};$$

$$\therefore 17 : 1 :: 1 : \frac{1}{17}, \therefore x = 2{\cdot}05 \text{ nearly.}$$

Again, let 2 and 2·05 be put for x, then

$$2{\cdot}05^3 - 2\,(2{\cdot}05) = 4{\cdot}515$$

$$2^3 - 4 \qquad = 4$$

$$\cdot 515 \text{ diff. of results};$$

$$\therefore \cdot 515 : \cdot 05 :: 1 : \cdot 0\dot{9}7,$$

$$\therefore x = 2{\cdot}097 \text{ nearly.}$$

Lastly, using this and the preceding approximate value of x, we are led to the proportion

$$\cdot 5122 : \cdot 047 :: \cdot 027 : \cdot 00247.$$

And as 2·097 is found upon trial to be too great, the last correction is subtractive,

$$\therefore x = 2{\cdot}097 - \cdot 00247 = 2{\cdot}09453.$$

The *true* value of the root x as far as five places of decimals is 2·09455, as found by the more easy and accurate method of approximation given in the Treatise on the 'General Solution of Equations,' which method entirely supersedes that of double position, when the equation to be solved is, like that above, in the ordinary rational form. But when a troublesome reduction is

necessary to bring the equation to such a form, and when moreover, as usually happens in such a case, the final reduced equation is likely to reach to a high degree, the above mode of solution may be advantageously employed. The following is an example of this kind:

2. Given $\sqrt[3]{(7x^3+4x^2)}+\sqrt{(20x^2-10x)}=28$, to find an approximate value of x.

Upon trial, x is found to lie between 4 and 5. The former supposition gives for result 24·733; and the latter, 31·129; therefore, taking the difference between these results and of the corresponding suppositions, as also the difference between 28 and the first result, we have by the rule,

$$6{\cdot}396 : 1 :: 3{\cdot}267 : {\cdot}51, \therefore x = 4{\cdot}51.$$

Actually substituting this for x, we find the value rather too small. Hence, repeating the operation with 4·51 and 4·6 we get $x=4{\cdot}51066$, which is true in all its decimals.

3. Given $x^3-x^2-2x=-1$, to find a near value of x.

The *true* value is 1·8019377.

4. Given $(\frac{1}{5}x^2-15)^2+x\sqrt{x}=90$, to find an approximate value of x.

Ans. 10·594831.

COMPOUND INTEREST AND ANNUITIES.

(149.) Interest is a certain sum paid for the use of money for any stated period. When the interest of this money is regularly received, the money, or principal, is said to be at simple interest; but when, instead of being regularly received, it is allowed to go to the increase of the principal, then the interest of the whole is called compound interest.

(150.) An annuity is a yearly income, or pension.

(151.) The present value of an annuity is that sum which, if put out at compound interest, shall be found just sufficient to pay the annuity as it becomes due.

(152.) PROBLEM 1. To find the amount of a given sum in any number of years at compound interest.

Let r represent the interest of 1*l.* for 1 year, and put $1l. + r = R$ $=$ the amount in 1 year.

Then $1l. : R :: R : R^2 =$ the amount in 2 years,

$1l. : R :: R^2 : R^3 =$ the amount in 3 years,

&c.

Therefore R^n is the amount of 1*l.* in n years, and consequently, the amount of £p is pR^n, $\therefore$ calling the amount a, we have

$$\log a = \log p + n \log R, \therefore \log p = \log a - n \log R.$$

Cor. 1. $$\log R = \frac{\log a - \log p}{n}, \text{ and } n = \frac{\log a - \log p}{\log R}.$$

Therefore any one of the quantities a, p, R, n, may be found from having the others given.

Cor. 2. If $a = mp$, then

$$n = \frac{\log mp - \log p}{\log R} = \frac{\log m + \log p - \log p}{\log R} = \frac{\log m}{\log R}.$$

Cor. 3. The value of p deduced from the equation $a = pR^n$, viz. $p = \frac{a}{R^n}$, and which is already expressed in logarithms above, is obviously no other than the present value of a sum a, receivable n years hence; interest being at the rate r for £1.

(153.) If the interest, instead of being due yearly, is supposed to become due half-yearly, quarterly, or after any other given period, then n, of course, instead of representing years, represents the proposed number of those periods, r being the interest for one period.

EXAMPLES.

1. How much would 300*l.* amount to in 4 years, at 4 per cent. per annum compound interest?

Here $p = 300$, $\text{R} = 1 + \frac{4}{100} = 1{\cdot}04$ and $n = 4$;

$\therefore \log a = \log p + n \log \text{R} = \log 300 + 4 \log 1{\cdot}04 = 2{\cdot}5452545$;

the number answering to which in the tables is 350·9575, ∴ the amount is 350*l.* 19*s.* 1¾*d.*

2. How much money must be placed out at compound interest to amount to 1000*l.* in 20 years, the interest being 5 per cent.?

Here $a = 1000$, $\text{R} = 1 + \frac{5}{100} = 1{\cdot}05$, and $n = 20$;

$\therefore \log p = \log a - n \log \text{R} = \log 1000 - 20 \log 1{\cdot}05, = 2{\cdot}576214$, the number answering to which is 376·89:

∴ the principal is 376*l.* 17*s.* 9½*d.*

3. At what interest must 300*l.* be placed out to amount to 350*l.* 19*s.* 2*d.* in 4 years?

Here $p = 300$, $a = 350{\cdot}957$, and $n = 4$;

$$\therefore \log \text{R} = \frac{\log a - \log p}{n} = \frac{\log 350{\cdot}957 - \log 300}{n} = {\cdot}0170333,$$

the number answering to which is 1·04:

$\therefore r = {\cdot}04$, and ${\cdot}04 \times 100 = 4$, the rate per cent.

4. In how many years will 400*l.* amount to 540*l.* at 4 per cent. compound interest?

Here $p = 400$, $a = 540$, and $\text{R} = 1 + \frac{4}{100} = 1{\cdot}04$:

$$\therefore n = \frac{\log a - \log p}{\log \text{R}} = \frac{\log 540 - \log 400}{\log 1{\cdot}04} = \frac{{\cdot}1303338}{{\cdot}0170333} =$$

7·65 years.

5. What will 600*l.* amount to in 6 years at 4½ per cent. compound interest, supposing the interest to be receivable half-yearly?

Here $p = 600$, $n = 12$, and $\text{R} = 1 + \frac{2{\cdot}25}{100} = 1{\cdot}0225$;

$\therefore \log a = \log p + n \log \text{R} = \log 600 + 12 \log 1{\cdot}0225 = 2{\cdot}8941109$; the number answering to which is 783·63;

∴ the amount is 783*l.* 12*s.* 7*d.*

6. In what time will a sum of money double itself at 5 per cent. compound interest?

Here $m=2$, and $\text{R}=1{\cdot}05$;

$$\therefore n=\frac{\log m}{\log \text{R}}=\frac{\log 2}{\log 1{\cdot}05}=\frac{{\cdot}3010300}{{\cdot}0211893}=14{\cdot}206=14\tfrac{1}{5} \text{ years nearly.}$$

7. In what time will 500*l.* amount to 900*l.* at 5 per cent. compound interest?

Ans. in 12·04 years.

8. What would 200*l.* amount to, if placed out for 7 years at 4 per cent. compound interest?

Ans. 263*l.* 3*s.* 8½*d.*

9. At what rate of compound interest must 376*l.* 17*s.* 9*d.* be placed out to amount to 1000*l.* in 20 years?

Ans. 5 per cent.

10. In what time will a sum of money double itself at 3½ per cent. compound interest?

Ans. 20·149 years.

Problem ii. To find the amount when the principal is increased not only by the interest, but also by some other sum at the same time.

The amount of the original principal p in n years is $p\text{R}^n$, and if A be the sum that is continually added, the first A will be at interest $n-1$ years; the second will be at interest $n-2$ years, &c., and therefore the sum of their amounts is

$$\text{AR}^{n-1}+\text{AR}^{n-2}+\ldots\ldots\text{AR}^{n-n}, \text{ or}$$
$$\text{A}(\text{R}^{n-1}+\text{R}^{n-2}+\ldots\ldots 1).$$

Now the terms within the parenthesis form a geometrical progression, whose last term is R^{n-1}, and ratio R, therefore the sum will be $\text{A}\times\frac{\text{R}^n-1}{\text{R}-1}$; $\therefore$ the whole amount is $p\text{R}^n+\text{A}\times\frac{\text{R}^n-1}{r}$, or, when $\text{A}=p$, then $a=p\text{R}^n+p\,\frac{\text{R}^n-1}{\text{R}-1}=p\,\frac{\text{R}^{n+1}-1}{r}$.

If, however, A is not added the nth year, then we have $a=p\text{R}^n+\text{AR}\,\frac{\text{R}^{n-1}-1}{r}$, or, when $\text{A}=p$, $a=p\text{R}\,\frac{\text{R}^n-1}{r}$.

Cor. 1. If, instead of $p = A$, we have $p = 0$, then $a = A\,\frac{R^n - 1}{r}$; which expresses the amount of an annuity A, at compound interest, left unpaid for n years.*

Cor. 2. If P be the present value of the annuity A for n years, P must be such, that if it were put out at compound interest for n years, it would amount to the same sum as the annuity, that is, we must have $PR^n = \frac{A(R^n - 1)}{r}$, whence $P = \frac{A}{r}\left\{1 - \frac{1}{R^n}\right\}$.

Hence, the present value is easily found from the tables of the present value of £1, receivable n years hence, which by (Cor. 3, p. 232,) is $\frac{1}{R^n}$.

Cor. 3. If n be infinite, then $\frac{1}{R^n}$ will vanish, in which case we shall have $P = \frac{A}{r}$, the present value of a *perpetuity* of £A per annum; that is, the present value of an annuity, to continue for ever, is found by dividing the annuity by the interest of £1 for a year.

If the annuity be in *reversion*, that is, not receivable till b years have elapsed, then the present value will be equal to the present value of the same annuity in *possession* for $b + n$ years, minus the present value of it for b years; that is,

$$P = \frac{A}{r}\left\{1 - \frac{1}{R^{b+n}}\right\} - \frac{A}{r}\left\{1 - \frac{1}{R^b}\right\}$$

$$= \frac{A}{r}\left\{\frac{1}{R^b} - \frac{1}{R^{b+n}}\right\} = \frac{1}{R^b}\,\frac{A}{r}\left\{1 - \frac{1}{R^n}\right\};$$

---

* If the rate per cent. be required, the annuity, number of years, and amount being given, we shall have to determine R from the above formula: that is, from the equation

$$\frac{R^n - 1}{R - 1} = \frac{a}{A};$$

or, page 99, from

$$R^{n-1} + R^{n-2} + R^{n-3} + \ldots . 1 = \frac{a}{A};$$

and this may be accomplished, when n is large, either by trial, as is usually recommended by writers on annuities, or by the more accurate methods delivered in the treatise on the *General Theory and Solution of Equations.*

therefore (*Cor.* 2) the present value may be found by multiplying the value of the annuity for n years by the present value of 1*l.* receivable at the end of b years.

EXAMPLES.

1. Suppose 300*l.* be put out at compound interest, and that to the stock is yearly added 20*l.*, what will be the amount at the expiration of 6 years, the interest being 4 per cent.?

$$\text{Here } p = 300,\ \text{A} = 20, \text{ and } r = \cdot 04,$$

$$\therefore a = p\text{R}^n + \frac{\text{AR}(\text{R}^{n-1} - 1)}{r} = 300(1\cdot 04)^6 + \frac{20 \times 1\cdot 04\,[(1\cdot 04)^5 - 1]}{\cdot 04}.$$

Now, $\log 300(1\cdot 04)^6 = \log 300 + 6 \log 1\cdot 04 = 2\cdot 5793211$

$= \log 379\cdot 595,$

and $\log (1\cdot 04)^5 = 5 \log 1\cdot 04 = \cdot 0851665 = \log 1\cdot 216652$:

$\therefore a = 379\cdot 595 + 500 \times 1\cdot 04 \times \cdot 216652 = 492\cdot 254 = 492$*l.* 5*s.* 1*d.*

2. How much will an annuity of 50*l.* amount to in 20 years at $3\frac{1}{2}$ per cent. compound interest?

$$\text{Here A} = 50,\ r = \frac{3\cdot 5}{100} = \cdot 035, \text{ and } n = 20,$$

$$\therefore a = \frac{\text{A}(\text{R}^n - 1)}{r} = \frac{50(1\cdot 035^{20} - 1)}{\cdot 035};$$

now $\log (1\cdot 035)^{20} = 20 \log 1\cdot 035 = \cdot 298806 = \log 1\cdot 989784$:

$$\therefore a = \frac{50 \times \cdot 989784}{\cdot 035} = 1413l.\ 19s.\ 7d.$$

3. Required the present value of an annuity of 50*l.* which is to continue 20 years at $3\frac{1}{2}$ per cent. compound interest.

By the last question, the amount is 1413*l.* 19*s.* 7*d.*, also $\text{R} = 1\cdot 035$, and $n = 20$:

$\therefore \text{PR}^n = 1413\cdot 98$, $\therefore \log \text{P} = \log 1413\cdot 98 - n \log \text{R} = 2\cdot 8516372 =$

$\log 710\cdot 62 = 710$*l.* 12*s.* 4*d.*

4. If the annual rent of a freehold estate be £A, what is its present value at 5 per cent. compound interest?

Here since n is infinite, $P = \frac{A}{r} = \frac{A}{\cdot 05} = 20A$; that is, the present value is 20 years' purchase.

5. What is the amount of an annuity of 30*l.* forborne 16 years, at $4\frac{1}{2}$ per cent. compound interest?

Ans. 681*l.* 11*s.* 7*d.*

6. In what time will an annuity of 20*l.* amount to 1000*l.* at 4 per cent. compound interest?

Ans. 28 years.

7. What is the present value of a perpetual annuity of £A, allowing 3 per cent. compound interest?

Ans. $33\frac{1}{3}$A.

(154.) We shall conclude this chapter on the application of logarithms with the following problem.

8. Suppose the interest of £1 for the xth part of a year to be $\frac{r}{x}$, it is required to determine the amount of £a when x is infinitely great.

Calling the amount A, we have

$$A = a\left(1 + \frac{r}{x}\right)^x$$

and taking the Napierian logarithms of each side of this equation,

$$\log A = \log a + x \log\left\{1 + \frac{r}{x}\right\}$$

$$= \log a + x\left\{\frac{r}{x} - \frac{r^2}{2x^2} + \frac{r^3}{3x^3} - \&c.\right\}$$

$$= \log a + r - \frac{r^2}{2x} + \frac{r^3}{3x^2} - \&c.$$

Let now x be infinitely great, then the terms having x in the denominators vanish, so that

$$\log A = \log a + r = \log ae^r,$$

$$\therefore A = ae^r,$$

that is, the amount is equal to a times the number whose Napierian logarithm is r. (See note A at the end.)

SCHOLIUM.

The preceding articles relate to *Annuities Certain*, that is to say, such as are independent of all risk or contingency. But another and most important department of the subject is that of *Life Annuities*, or those which continue payable only during the existence of one or more lives.

It is evident that all computations in reference to this latter kind of annuities must involve conditions from which the calculations above are necessarily free; conditions dependent upon the duration of human life, and consequently implying risk and uncertainty. But whatever may be the amount of this uncertainty in any individual case, yet long and careful observation has shown that when large numbers of persons of the same age are viewed in the aggregate, the average rate of mortality is subject to a certain fixed and uniform law, in reference to which the chance that each individual has of surviving any proposed number of years may be accurately calculated, or the average value of each entire life determined. From this determination a great variety of most important questions relative to the value of human life, or to any combination of lives, may be solved; and thence the values of annuities dependent upon these contingencies may be fairly and safely estimated. The investigation of such questions requires an application of the general *theory of probabilities*, a brief sketch of which will be given in the APPENDIX to the present work.

CHAPTER VIII.

ON SERIES.

THE DIFFERENTIAL METHOD.

(155.) The Differential Method is the method of finding the successive differences of the terms of a series, and thence any advanced term, or the sum of the whole series. It is of considerable use in the construction of tables of numbers, as it enables us, after a few consecutive terms of such a table are computed, to carry on the series by easy and uniform operations, involving addition and subtraction only.

PROBLEM I.

(156.) To find the first term of any order of differences.

Let $a, b, c, d, e,$ &c. represent any series; then, if the successive differences of the terms be taken, these differences will form a new series, which is called the *first order of differences*; in like manner, if the successive differences of the terms of this last series be taken, a new series, called the *second order of differences*, will be obtained, &c. Thus,

1st order of differences,

$$\begin{array}{llllll}
b-a, & c-b, & d-c, & & e-d, & \&c. \\
 & \underline{b-a} & \underline{c-b} & & \underline{d-c} & \\
\text{2d order, } & c-2b+a, & d-2c+b, & & e-2d+c, & \&c. \\
 & & \underline{c-2b+a} & & \underline{d-2c+b} & \\
\text{3d order} & \ldots\ldots & d-3c+3b-a, & & e-3d+3c-b, & \&c. \\
\&c. & & & & \&c. &
\end{array}$$

Now, since in the first order the first term in any difference is the same, except the sign, as the second in the succeeding difference, in subtracting any difference from the succeeding, the first term of the former must be placed under the second term of the latter, and, consequently, the same must take place in every succeeding order.

Hence the coefficients of the several terms composing either of the differences belonging to any order are respectively the same as the coefficients of the terms in the expanded binomial, being generated exactly in the same way,* the terms that are subtracted being in reality added with contrary signs.

Therefore, representing the *first difference* of the 1st, 2d, 3d, &c. order respectively by Δ^1, Δ^2, Δ^3, *&c.*, we have, for the first difference of the nth order,

When n *is an even number,*

$$\Delta^n = a - nb + \frac{n(n-1)}{2}c - \frac{n(n-1)(n-2)}{2.3}d + \text{\&c.}$$

When n *is an odd number,*

$$\Delta^n = -a + nb - \frac{n(n-1)}{2}c + \frac{n(n-1)(n-2)}{2.3}d - \text{\&c.}$$

---

* Thus,

$$\begin{array}{l}
1-1 = \text{coefficients of the first order,} \\
1-1 \\
\hline
1-1 \\
\quad -1+1 \\
\hline
1-2+1 = \text{coefficients of the second order,} \\
1-1 \\
\hline
1-2+1 \\
\quad -1+2-1 \\
\hline
1-3+3-1 \quad \ldots\ldots\ldots \text{ third order,} \\
\qquad \text{\&c.} \qquad\qquad\qquad\quad \text{\&c.}
\end{array}$$

EXAMPLES.

1. Required the first term of the fourth order of differences of the series 1, 8, 27, 64, 125, &c.

Here a, b, c, d, e, &c. $= 1, 8, 27, 64, 125$, &c. and $n = 4$;

$$\therefore a - nb + \frac{n(n-1)}{2}c - \frac{n(n-1)(n-2)}{2.3}d +$$

$$\frac{n(n-1)(n-2)(n-3)}{2.3.4}e = a - 4b + 6c - 4d + e = 1 - 32 +$$

$$162 - 256 + 125 = 0;$$

hence the first term of the fourth order is 0.

2. Required the first term of the fifth order of differences of the series $1, 3, 3^2, 3^3, 3^4$, &c.

Here a, b, c, d, e, &c. $= 1, 3, 9, 27, 81$, &c. and $n = 5$,

$$\therefore -a + nb - \frac{n(n-1)}{2}c + \frac{n(n-1)(n-2)}{2.3}d -$$

$$\frac{n(n-1)(n-2)(n-3)}{2.3.4}e + \text{\&c.} = -a + 5b - 10c + 10d - 5e +$$

$f = -1 + 15 - 90 + 270 - 405 + 243 = 32 =$ the first term of the fifth order of differences.

3. Required the first term of the third order of differences of the series $1, 2^3, 3^3, 4^3$, &c.

Ans. 6.

4. Required the first term of the fourth order of differences of the series 1, 6, 20, 50, 105, &c.

Ans. 2.

PROBLEM II.

(157.) To find the nth term of the series a, b, c, d, e, &c.

Let Δ^1, Δ^2, Δ^3, Δ^4, &c. represent the first term in the first, second, third, fourth, &c. order of differences respectively; then, in

the general expressions for the first term of the nth order, we shall have, by making n successively equal to 1, 2, 3, &c., and transposing,

$$
[A]\quad \begin{aligned}
b &= a + \Delta^1, \\
c &= -a + 2b + \Delta^2, \\
d &= a - 3b + 3c + \Delta^3, \\
e &= -a + 4b - 6c + 4d + \Delta^4, \\
f &= a - 5b + 10c - 10d + 5e + \Delta^5, \\
\&c. &= \&c.
\end{aligned}
$$

Or, by substitution,

$$
[B]\quad \begin{aligned}
b &= a + \Delta^1, \\
c &= a + 2\Delta^1 + \Delta^2, \\
d &= a + 3\Delta^1 + 3\Delta^2 + \Delta^3, \\
e &= a + 4\Delta^1 + 6\Delta^2 + 4\Delta^3 + \Delta^4, \\
f &= a + 5\Delta^1 + 10\Delta^2 + 10\Delta^3 + 5\Delta^4 + \Delta^5, \\
\&c. &= \&c.
\end{aligned}
$$

where the coefficients of a, Δ^1, Δ^2, Δ^3, &c. in the $n + 1$th term of the series a, b, c, d, &c. are the same as the coefficients of the terms of a binomial raised to the nth power, that is, the $n + 1$th term is

$$a + n\Delta^1 + \frac{n(n-1)}{2}\Delta^2 + \frac{n(n-1)(n-2)}{2.3}\Delta^3 + \&c.$$

and therefore the nth term is

$$a + (n-1)\Delta^1 + \frac{(n-1)(n-2)}{2}\Delta^2 + \frac{(n-1)(n-2)(n-3)}{2.3}\Delta^3 + \&c.$$

EXAMPLES.

1. Required the tenth term of the series 1, 4, 8, 13, 19, &c.

1, 4, 8, 13, 19,

3, 4, 5, 6,

1, 1, 1,

0.

Here the first terms of the differences are 3, 1, and 0;

that is, $\Delta^1=3$, $\Delta^2=1$, and $\Delta^3=0$, also $a=1$, and $n=10$;

$$\therefore a+(n-1)\Delta^1+\frac{(n-1)(n-2)}{2}\Delta^2=1+27+36=64,$$

which is the tenth term required.

2. Required the twelfth term in the series $1^3, 2^3, 3^3, 4^3, 5^3$, &c.

$$\begin{array}{c} 1,\ 8,\ 27,\ 64,\ 125, \\ 7,\ 19,\ 37,\ 61, \\ 12,\ 18,\ 24, \\ 6,\ 6, \\ 0. \end{array}$$

Here $\Delta^1=7$, $\Delta^2=12$, $\Delta^3=6$, $\Delta^4=0$, also $a=1$, and $n=12$;

$$a+(n-1)\Delta^1+\frac{(n-1)(n-2)}{2}\Delta^2+\frac{(n-1)(n-2)(n-3)}{2\,.\,3}$$
$$\Delta^3=1+77+660+990=1728, \text{ the twelfth term.}$$

The above method is useful in enabling us to find a remote term in a series of which the differences at length become 0. But when such a series is to be regularly continued by help of a few of its leading terms, it will be best to find, by actual subtraction, the several orders of differences, till they at length become constant, and then, by the successive additions of each final difference to that in the order above, to ascend through these orders, thus introducing a new term into each, till we reach the original series. An example or two will fully illustrate our meaning.

3. Suppose it were required to construct a table of squares from a few of the leading terms of such a table, viz. 1, 4, 9, 16, &c. We shall have, by differencing,

$$\begin{array}{c} 1,\ 4,\ 9,\ 16, \\ 3,\ 5,\ 7, \\ 2,\ 2, \end{array}$$

So that the second differences are constant. Hence, adding 2 to 7, a new term 9 will be introduced into the second order; and adding this

9 to 16, an additional square, viz. 25, will be obtained; and by repeating the operation with the new terms thus found, the next square 36 will be introduced into the series, and so on, as far as we please.

In this business of forming a table by differences, something, as to facility, will depend upon arrangement. Thus, in the present case, the work might be conveniently carried on as follows:

2d diff.	2							
1st diff.	7	9	11	13	15	17	19	&c.
Squares	16	25	36	49	64	81	100	&c.

where the row of first differences is prolonged by adding the constant second difference, 2, to each in succession; and the row of squares is continued by adding to every new term in the first differences the last found square.

4. Again, let it be required to construct a table of cubes. Here, referring to the differences in example 2, above, and imitating the process just explained, the work will be as follows:

3d diff.	6							
2d diff.	24	30	36	42	48	54	&c.
1st diff.	61	91	127	169	217	271	
Cubes	125	216	343	512	729	1000	&c.

And in this manner, whenever the differences at length become constant, may a series of numbers be extended to any length by successive additions merely. Even when the differences do not become constant, but vary in every order, yet the variation may be so exceedingly small a decimal that the differences may be considered constant, and employed as such for the computation of a considerable number of new terms, without introducing into those terms any numerical error of consequence. In the computation of Tables of Logarithms, for instance, when the terms in any order of differences vary from each other only by a significant figure in the ninth or tenth place of decimals, a great many terms, computed upon the hypothesis that this variation is actually zero, may be determined with perfect accuracy, as far as *seven* places of decimals, the utmost extent to which our best modern tables are carried.

The formulas marked [A] are also useful in the business of forming such Tables, since they enable us very readily to interpolate any wanting term in the series, by the aid of those between which it is to be interposed. The following is an example of their use.

5. Given the logarithms of the numbers 101, 102, 104, and 105, to find the logarithm of 103.

Here, of five consecutive terms, a, b, c, d, e, four are given to find the intermediate one. To accomplish this with perfect accuracy would require us to know the value of Δ^4 in [A], which is itself not generally determinable without the term sought. But the four logarithms which are here given are themselves not strictly accurate, being indeed carried only to a limited number of decimals, usually seven, as before stated; and, from the slow increase of the logarithms at the part of the table where these occur, we may easily assure ourselves that Δ^4 can have no significant figure in the first seven places of decimals: it may therefore be rejected, without introducing error. Hence, regarding Δ^4 as 0, we have, for the determination of the term c sought, the equation

$$e = -a + 4b - 6c + 4d,$$

$$\therefore c = \frac{4(b+d) - (a+e)}{6},$$

which expression is thus calculated,

$$
\begin{aligned}
a = \log 101 &= 2{\cdot}0043214 \\
b = \log 102 &= 2{\cdot}0086002 \\
d = \log 104 &= 2{\cdot}0170333 \\
e = \log 105 &= 2{\cdot}0211893 \\
\therefore 4(b+d) &= 16{\cdot}1025340 \\
(a+e) &= 4{\cdot}0255107 \\
6)\ &\ 12{\cdot}0770233 \\
c = \log 103 &= 2{\cdot}0128372
\end{aligned}
$$

And in this manner may any intermediate term in a series be calculated, provided always that, p being the number of given terms, the difference Δ^p may be rejected, without committing sensible error.

The student who wishes for further information upon this subject of *interpolation*, more especially in reference to its utility in computing logarithms, may consult Chap. II of the 'Essay on the Computation of Logarithms,' before referred to.

6. Required the twentieth term of the series 1, 3, 5, 7, &c.

Ans. 39.

7. Required the twentieth term of the series 1, 3, 6, 10, 15, &c.

Ans. 210.

8. Required the fifteenth term of the series $1, 2^2, 3^2, 4^2$, &c.

Ans. 225.

9. Given the logarithms of 50, 51, 52, 54, and 55, to find the logarithm of 53.

Ans. $\log 53 = 1{\cdot}7242759$.

PROBLEM III.

(158.) To find the sum of n terms of a series.

Let the proposed series be, as before, a, b, c, d, &c.; then, by means of the general expressions in last Problem, we shall be able to find the sum of n terms of this series, provided we can devise another series, such that either the $n+1$th or the nth term may always be equal to n terms of the proposed. Now the series whose $n+1$th term equals the sum of n terms of the proposed at once presents itself; it is the series

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d,\ \&c.$$

of which the first differences, viz.

$$a,\quad b,\quad c,\quad d,\quad \&c.$$

form the original series; and, consequently, that which is Δ^1 in the new series is the first term in the proposed, and that which is Δ^2 in the former is Δ^1 in the latter, and so on. Hence, referring

to the general expression in last problem, we have for the $n+1$th term of the new series, that is, for the sum of n terms of the proposed, the formula

$$S = na + \frac{n(n-1)}{2}\Delta^1 + \frac{n(n-1)(n-2)}{2 \cdot 3}\Delta^2 + \&c.$$

EXAMPLES.

1. Required the sum of n terms of the series 1, 3, 5, 7, &c.

$$\begin{array}{c} 1,\ 3,\ 5,\ 7, \\ 2,\ 2,\ 2, \\ 0,\ 0, \end{array}$$

Here $\Delta^1 = 2$, and $\Delta^2 = 0$, also $a = 1$;

$$\therefore na + \frac{n(n-1)}{2}\Delta^1 = n^2 = \text{sum of } n \text{ terms.}$$

2. Required the sum of n terms of the series $1, 2^2, 3^2, 4^2, 5^2$, &c.

$$\begin{array}{c} 1,\ 4,\ 9,\ 16,\ 25, \\ 3,\ 5,\ 7,\ 9, \\ 2,\ 2,\ 2, \\ 0,\ 0. \end{array}$$

Here, $\Delta^1 = 3$, $\Delta^2 = 2$, and $\Delta^3 = 0$, also $a = 1$;

$$\therefore na + \frac{n(n-1)}{2}\Delta^1 + \frac{n(n-1)(n-2)}{2 \cdot 3}\Delta^2 = \frac{2n + 3n^2 - 3n}{2} +$$

$$\frac{n^3 - 3n^2 + 2n}{3} = \frac{n(n+1)(2n+1)}{6} = \text{sum of } n \text{ terms.}$$

It may be remarked, that this is the expression for the number of shot in a pyramidal pile, having a square base of n shot in each side. In like manner, the answer to Example 5, following, gives the number of shot in the pile when the base is a triangle, having n shot in each side.

3. Required the sum of n terms of the series 1, 2, 3, 4, 5, &c.

Ans. $\frac{n(n+1)}{2}$.

4. Required the sum of twelve terms of the series 1, 4, 8, 13, 19, &c.
Ans. 430.

5. Required the sum of n terms of the series 1, 3, 6, 10, 15, &c.
Ans. $\frac{n(n+1)(n+2)}{}$.

6. Required the sum of n terms of the series $1 + 6 + 15 + 28 +$ &c.
Ans. $n + \frac{5n(n-1)}{2} + \frac{2n(n-1)(n-2)}{3}$.

7. Required the sum of n terms of the series $1, 2^3, 3^3, 4^3$, &c.
Ans. $\frac{n^2(n+1)^2}{4}$.*

8. Required the sum of n terms of the series $1, 2^4, 3^4, 4^4$, &c.
Ans. $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$.

---

* Comparing this with the answer to example 3, we find that $1 + 2^3 + 3^3 +$ &c. $= (1 + 2 + 3 +$ &c.$)^2$; that is, the sum of the cubes of any number of terms of the series 1, 2, 3, &c. is equal to the square of their sum; a property which was discovered by Peletarius, and published in 1558. (See Powell's History of Natural Philosophy, p. 123.)

The following properties of this series, and of that in example 5, seem also to deserve notice.

$$2^2,\ 3^2,\ 4^2,\ 5^2,\ 6^2,\ \text{\&c.}$$
$$= 3 + 1,\ 6 + 3,\ 10 + 6,\ 15 + 10,\ 21 + 15,\ \text{\&c.}$$

And

$$2^3,\ 3^3,\ 4^3,\ 5^3,\ 6^3,\ \text{\&c.}$$
$$= 3^2 - 1^2,\ 6^2 - 3^2,\ 10^2 - 6^2,\ 15^2 - 10^2,\ 21^2 - 15^2,\ \text{\&c.}$$
$$= (3-1)^3,\ (6-3)^3,\ (10-6)^3,\ (15-10)^3,\ (21-15)^3,\ \text{\&c.}$$

Consequently, in the series

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ \text{\&c.}$$

the sum of every two consecutive numbers is a square number, and the difference of their squares the cube of that number, which number is the difference of the two employed. And from these properties may a table of squares and cubes be readily constructed.

ON THE SUMMATION OF INFINITE SERIES.

(159.) An Infinite Series is a progression of quantities proceeding onwards without termination, but usually according to some regular law discoverable from a few of the leading terms.

(160.) A converging series is a series whose successive terms decrease or become less and less as the series

$$\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \&c.$$

x being any number greater than unit. The *finite quantity* to which we continually approach, by summing up more and more of the leading terms, is the quantity to which the series converges, and to which it actually attains only when taken in all its infinitude of terms. Should the series be infinite in value, as well as in extent, it is not regarded as convergent, even though its terms successively diminish. The series $0 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$, for instance, is not considered to be convergent, as it does not tend to any limit, its value being infinite, since (139) the Napierian log of 0 is $-(1 + \frac{1}{2} + \frac{1}{3} + \&c.)$, and this logarithm we know to be $-\infty$.

(161.) A diverging series is one whose successive terms increase or become greater and greater; such is the series

$$\frac{1}{1+2} = 1 - 2 + 4 - 8 + 16 - \&c.$$

(162.) A neutral series is one whose terms are all equal, but have signs alternately + and —, as the series

$$\frac{1}{1+1} = 1 - 1 + 1 - 1 + 1 - \&c.*$$

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* For some remarks on this series, see the article on Indeterminate Coefficients, towards the end of the present chapter (page 275).

(163.) An ascending series is one in which the powers of the unknown quantity ascend, as in the series

$$a + bx + cx^2 + dx^3 + \&c.$$

(164.) A descending series is one in which the powers of the unknown quantity descend, as in the series

$$a + bx^{-1} + cx^{-2} + dx^{-3} + \&c.$$

(165.) The summation of series is the finding a finite expression equivalent to the series; or the algebraic form from the development of which the series is produced.

(166.) As different series are often governed by very different laws, the methods of finding the sum which are applicable to one class of series will not apply universally; a great variety of useful series may be summed by help of the following obvious considerations:

(167.) I. Since

$$\frac{q}{n} - \frac{q}{n+q} = \frac{pq}{n(n+p)}, \therefore \frac{q}{n(n+p)} = \frac{1}{p}\left\{\frac{q}{n} - \frac{q}{n+p}\right\};$$

that is, any fraction of the form $\frac{q}{n(n+p)}$ is equal to $\frac{1}{p}$th the difference between the two fractions $\frac{q}{n}$ and $\frac{q}{n+p}$; hence, if this difference be known, the value of $\frac{q}{n(n+p)}$ will be known, whether $\frac{q}{n}$ and $\frac{q}{n+p}$ be known or not; and it therefore follows, that if there be any series of fractions, each having the form $\frac{q}{n(n+p)}$, the sum of the series will be equal to $\frac{1}{p}$th the difference between a series of fractions of the form $\frac{q}{n}$, and another of the form $\frac{q}{n+p}$, and, if this difference can be obtained, the sum of the proposed series may be readily found, whatever be the values of p, q, and n.

EXAMPLES.

1. Required the sum of the series $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} +$ &c. continued to infinity.

Here $q = 1$, and $p = 1$, also $n = 1, 2, 3$, &c. successively;

$$\therefore \left\{ \begin{array}{l} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{\&c. ad inf.} \\ -(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{\&c. ad inf.}) \end{array} \right\} = 1 = \text{sum.}$$

2. Required the sum of the above series to n terms,

$$\left\{ \begin{array}{l} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots \frac{1}{n} \\ -(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots \frac{1}{n} + \frac{1}{n+1}) \end{array} \right\} = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

3. Required the sum of the series $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} +$ &c. ad infinitum.

Here $p = 2$,

$$\left\{ \begin{array}{l} 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \text{\&c. ad inf.} \\ -(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \text{\&c. ad inf.}) \end{array} \right\} = 1, \therefore \frac{1}{p} = \frac{1}{2} = \text{sum.}$$

4. Required the sum of the above series to n terms.

$$\left\{ \begin{array}{l} 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \frac{1}{2n-1} \\ -(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \frac{1}{2n-1} + \frac{1}{2n+1}) \end{array} \right\} = 1 - \frac{1}{2n+1} = \frac{2n}{2n+1},$$

$$\text{and } \frac{1}{p}\text{th of this is } \frac{n}{2n+1} = \text{sum.}$$

5. Required the sum of the series $\frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 6} + \frac{1}{4 \cdot 7}$ + &c. to infinity.

Here $p = 3$,

$$\left\{\begin{matrix} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c. \\ -(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \&c., \end{matrix}\right\} = 1 + \frac{1}{2} + \frac{1}{3} = 1\tfrac{5}{6},$$

and $\frac{1}{p}$th of this is $\frac{11}{18}$ = sum.

6. Required the sum of n terms of the above series,

$$\left\{\begin{matrix} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdot\cdot \frac{1}{n} \\ -(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdot\cdot \frac{1}{n+3}) \end{matrix}\right\} = 1 + \frac{1}{2} + \frac{1}{3} -$$

$$\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right) = \frac{n}{n+1} + \frac{n}{2n+4} + \frac{n}{3n+9},$$

$$\therefore \frac{n}{3n+3} + \frac{n}{6n+12} + \frac{n}{9n+27} = \text{sum}.$$

7. Required the sum of the series $\frac{2}{3\,.\,5} - \frac{3}{5\,.\,7} + \frac{4}{7\,.\,9} - \frac{5}{9.11} + \&c.$

Here $p = 2$, and $q = 2, 3, 4,$ &c. successively;

$$\left\{\begin{matrix} \frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} + \&c. \\ -(\frac{2}{5} - \frac{3}{7} + \frac{4}{9} - \&c.) \end{matrix}\right\} = \frac{2}{3} - 1 + 1 - 1 + 1 - 1 + \&c. = \frac{2}{3} - \frac{1}{2} = \frac{1}{6},$$

and $\frac{1}{p}$ of this is $\frac{1}{12}$ = sum.

8. Required the sum of the series $1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \&c.$ ad infinitum.

This series is evidently the same as the following, viz.

$$1 + \frac{1}{3} + \frac{1}{2.3} + \frac{1}{2.5} + \&c.$$

and dividing by 2, it becomes

$$\frac{1}{2} + \frac{1}{2\,.\,3} + \frac{1}{3\,.\,4} + \frac{1}{4\,.\,5} + \&c.$$

whose sum is 1 (Ex. 1st); $\therefore$ the sum of the proposed series is 2.

9. Required the sum of the series $\frac{1}{3.8} + \frac{1}{6.12} + \frac{1}{9.16} + \&c.$ ad infinitum.

This series is the same as $\frac{1}{4}\left(\frac{1}{3.2}+\frac{1}{6.3}+\frac{1}{9.4}+\text{\&c.}\right)$

$$=\frac{1}{12}\left(\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\text{\&c.}\right)=(\text{Ex. }1)\ \frac{1}{12};$$

also the sum to n terms is $\frac{n}{12\,(n+1)}$.

10. Required the sum of the series $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+$ &c. ad infinitum.

Ans. $\frac{3}{4}$.

11. Required the sum of the series $\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-$ &c. ad infinitum.

Ans. $\frac{1}{4}$.

12. Required the sum of the series in example 7 to n terms.

Ans. $\frac{1}{12}\pm\frac{1}{4\,(2n+3)}$, according as n is odd or even.

13. Required the sum of the series $\frac{4}{1.5}+\frac{4}{5.9}+\frac{4}{9.13}+\frac{4}{13.17}+$ &c. ad infinitum.

Ans. 1.

(168.) II. Also, since

$$\frac{q}{n(n+p)}-\frac{p}{(n+p)(n+2p)}=\frac{2pq}{n(n+p)(n+2p)},\ \therefore \frac{q}{n(n+p)(n+2p)}$$
$$=\frac{1}{2p}\left\{\frac{q}{n\,(n+p)}-\frac{q}{(n+p)\,(n+2p)}\right\};$$

hence the sum of any series of fractions, each of which is of the form $\frac{q}{n(n+p)(n+2p)}$, is equal to $\frac{1}{2p}$ the difference between one series, whose terms are of the form $\frac{q}{n\,(n+p)}$, and another whose terms are of the form $\frac{q}{(n+p)\,(n+2p)}$.

EXAMPLES.

1. Required the sum of the series $\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}+$ &c. ad infinitum.

Here $p=1$, and $q=4, 5, 6,$ &c. successively;

$$\therefore \left\{\begin{array}{l}\frac{4}{1.2}+\frac{5}{2.3}+\frac{6}{3.4}+\text{\&c.} \\ \quad -\left(\frac{4}{2.3}+\frac{5}{3.4}+\text{\&c.}\right)\end{array}\right\}=$$

$\frac{4}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+$ &c. (Art. 167, ex. 1) $=2\frac{1}{2}$, and $\frac{1}{2p}$ of this is $1\frac{1}{4}$ $=$ sum.

2. Required the sum of $\frac{3}{5.8.11}+\frac{9}{8.11.14}+\frac{15}{11.14.17}+$ &c. ad infinitum.

Here $p=3$,

$$\left\{\begin{array}{l}\frac{3}{5.8}+\frac{9}{8.11}+\frac{15}{11.14}+\text{\&c.} \\ \quad -\left(\frac{3}{8.11}+\frac{9}{11.14}+\text{\&c.}\right)\end{array}\right\}=$$

$$\frac{3}{5.8}+\frac{6}{8.11}+\frac{6}{11.14}+\text{\&c.}=$$

$\frac{3}{5.8}+\frac{1}{3}\left\{\begin{array}{l}\frac{6}{8}+\frac{6}{11}+\frac{6}{14}+\text{\&c.} \\ -\left(\frac{6}{11}+\frac{6}{14}+\text{\&c.}\right)\end{array}\right\}=\frac{3}{5.8}+\frac{2}{8}=\frac{13}{40}$; and $\frac{1}{2p}$ of this is $\frac{13}{240}=$ sum.

3. Required the sum of the series $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+$ &c. ad infinitum.

Ans. $\frac{1}{8}$.

4. Required the sum of the series $\frac{1}{1.3.5}+\frac{4}{3.5.7}+\frac{7}{5.7.9}+\frac{10}{7.9.11}+$ &c. ad infinitum.

Ans. $\frac{5}{24}$.

5. Required the sum of the series

$$\frac{a}{n(n+p)(n+2p)}+\frac{a+b}{(n+p)(n+2p)(n+3p)}+\frac{a+2b}{(n+2p)(n+3p)(n+4p)} \text{ \&c. ad infinitum.}$$

Ans. $\frac{pa+bn}{2p^2n(n+p)}$.

(169.) III. Likewise, since

$$\frac{q}{n(n+p)(n+2p)}-\frac{q}{(n+p)(n+2p)(n+3p)}=\frac{3pq}{n(n+p)(n+2p)(n+3p)}, \therefore \frac{q}{n(n+p)(n+2p)(n+3p)}=$$

$$\frac{1}{3p}\left\{\frac{q}{n(n+p)(n+2p)}-\frac{q}{(n+p)(n+2p)(n+3p)}\right\};$$

therefore, any series of fractions, of the form $\frac{q}{n(n+p)(n+2p)(n+3p)}$, is equal to $\frac{1}{3p}$, the difference between a series of the form $\frac{q}{n(n+p)(n+2p)}$, and another of the form

$$\frac{q}{(n+p)(n+2p)(n+3p)}.$$

EXAMPLES.

1. Required the sum of the series $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+$ &c. ad infinitum.

Here $p=1$,

$$\left.\begin{array}{l} \dfrac{1}{1.2.3}+\dfrac{1}{2.3.4}+\dfrac{1}{3.4.5}+\&c. \\ \qquad -(\dfrac{1}{2.3.4}+\dfrac{1}{3.4.5}+\&c.) \end{array}\right\}=\frac{1}{1.2.3}=\tfrac{1}{6};$$

$$\therefore \frac{1}{3p}\left(\tfrac{1}{6}\right)=\tfrac{1}{18}=\text{sum.}$$

2. Required the sum of the series $\dfrac{1}{1.3.5.7}+\dfrac{2}{3.5.7.9}+\dfrac{3}{5.7.9.11}+$ &c. ad infinitum.

Here $p=2$,

$$\left.\begin{array}{l} \dfrac{1}{1.3.5}+\dfrac{2}{3.5.7}+\dfrac{3}{5.7.9}+\&c. \\ \qquad -(\dfrac{1}{3.5.7}+\dfrac{2}{5.7.9}+\&c.) \end{array}\right\}=$$

$$\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\&c.=\tfrac{1}{12};$$

$$\therefore \frac{1}{3p}\left(\tfrac{1}{12}\right)=\tfrac{1}{72}=\text{sum.}$$

3. Required the sum of the series $\dfrac{2}{3.6.9.12}+\dfrac{5}{6.9.12.15}+\dfrac{8}{9.12.15.18}+$ &c. ad infinitum.

Ans. $\frac{7}{2916}$.

4. Required the sum of the series $\dfrac{6^2}{1.2.3.4}+\dfrac{7^2}{2.3.4.5}+\dfrac{8^2}{3.4.5.6}+$ &c. ad infinitum.

Ans. $\frac{88}{35}$.

(170.) In a similar manner, it may be shown that the sum of any series of fractions of the form

$$\frac{q}{n(n+p)(n+2p)\ldots\ldots(n+mp)}$$

is equal to $\frac{1}{mp}$ the difference between a series of the form

$$\frac{q}{n(n+p)\ (n+2p)\ \ldots\ [n+(m-1)p]},$$

and another of the form

$$\frac{q}{(n+p)\ (n+2p)\ \ldots\ (n+mp)}.$$

(171.) Again, since

$$\frac{a(a+b)\,(a+2b)\ldots(a+pb)}{n(n+b)\ldots[n+(p-1)b]}-\frac{a(a+b)\,(a+2b)\ldots[a+(p+1)b]}{n(n+b)\ldots(n+pb)}$$

$$=\frac{a\,(n-a-b)\ (a+b)\ (a+2b)\ldots(a+pb)}{n\,(n+b)\ (n+2b)\ldots(n+pb)},$$

$$\therefore\frac{a\,(a+b)\ (a+2b)\ldots(a+pb)}{n\,(n+b)\,(n+2b)\ldots(n+pb)}=$$

$$\frac{1}{n-a-b}\left\{\frac{a\,(a+b)\ldots(a+pb)}{n\,(n+b)\ldots[n+(p-1)\,b]}-\right.$$

$$\left.\frac{a\,(a+b)\ldots[a+(p+1)\,b]}{n\,(n+b)\ldots(n+pb)}\right\}.$$

Hence, any series of fractions of the form

$$\frac{a\,(a+b)\ldots(a+pb)}{n\,(n+b)\ldots(n+pb)}$$

is equal to $\frac{1}{n-a-b}$, the difference of a series of the form

$$\frac{a\,(a+b)\ldots(a+pb)}{n\,(n+b)\ldots[n+(p-1)b]},$$

and another of the form

$$\frac{a\,(a+b)\ldots[a+(p+1)\,b]}{n\,(n+b)\ldots(n+pb)}.$$

EXAMPLES.

1. Required the sum of the series $\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+\frac{1.3.5.7}{2.4.6.8}$ + &c. to r terms.

Here $a=1$, $b=2$, and $n=2$,

$$\left\{\begin{array}{l} 1+\frac{1.3}{2}+\frac{1.3.5}{2.4}+\ldots\frac{1.3.5.7\ldots.(2r-1)}{2.4.6\ldots.(2r-2)} \\ -\left(\frac{1.3}{2}+\frac{1.3.5}{2.4}+\ldots\ldots\ldots\frac{1.3.5.7\ldots.(2r+1)}{2.4.6\ldots.2r}\right)\end{array}\right\}=$$

$$1-\frac{1.3.5.7\ldots..(2r+1)}{2.4.6\ldots.2r},$$

and $\frac{1}{n-a-b}$ of this is $\frac{1.3.5.7\ldots.(2r+1)}{2.4.6\ldots.2r}-1=$ sum of r terms; when r is infinite, this expression is evidently infinite also.

2. Required the sum of the series

$$\frac{a}{n}+\frac{a(a+b)}{n(n+b)}+\frac{a(a+b)(a+2b)}{n(n+b)(n+2b)}+\text{\&c. to } r \text{ terms},$$

$$\left\{\begin{array}{l} a+\frac{a(a+b)}{n}+\ldots\frac{a(a+b)\ldots.[a+(r-1)b]}{n(n+b)\ldots.[n+(r-2)b]} \\ -\left(\frac{a(a+b)}{n}+\ldots\ldots\ldots\frac{a(a+b)\ldots.(a+rb)}{n(n+b)\ldots.[n+(r-1)b]}\right)\end{array}\right\}$$

$$=a-\frac{a(a+b)(a+2b)\ldots.(a+rb)}{n(n+b)\ldots.[n+(r-1)b]};$$

$$\therefore \text{sum}=\frac{a}{n-a-b}-\frac{a(a+b)(a+2b)\ldots.(a+rb)}{(n-a-b)n(n+b)\ldots.[n+(r-1)b]}.$$

If r be infinite, then this expression for the sum will become definite only in particular cases. Thus, if $n=a+2b$, the second fraction in the above expression will be

$$\frac{a(a+b)}{b[a(r+1)b]},$$

which evidently vanishes when r is infinite, in which case the sum is $\frac{a}{n-a-b}$; the same fraction would, of course, vanish, if n were greater than $a+2b$. So that in these cases we should always have for the sum the definite result $\frac{a}{n-a-b}$.

But if n were equal to $a+b$, then the said fraction would become

$$\frac{a\,(a+b)\,(a+2b)\,\ldots\ldots\,(a+rb)}{0\,(a+b)\,(a+2b)\,\ldots\ldots\,(a+rb)}=\frac{a}{0}$$

and the sum would be $\frac{a-a}{0}=\frac{0}{0}$, an expression of no definite signification in its present form. The sum presents itself under the same indefinite form even when r is finite, provided $n=a+b$, as will appear by inspecting the general expression; in these cases therefore the sum is not determinable by the method here employed.

3. Required the sum of r terms of the series $\frac{2}{3}+\frac{2.4}{3.5}+\frac{2.4.6}{3.5.7}+\frac{2.4.6.8}{3.5.7.9}+$ &c.

Ans. $\frac{2.4.6.8\ldots\ldots(2r+2)}{3.5.7.9\ldots\ldots(2r+1)}-2.$

4. Required the sum of the series $\frac{2}{5.6}+\frac{2.3}{5.6.7}+\frac{2.3.4}{5.6.7.8}+$ &c. ad infinitum.

Ans. $\frac{2}{15}$.

(172.) The method of finding the summation of series delivered in the preceding articles, is capable of considerable extension: we shall however in this place adduce only one more instance of its application.

It is obvious that the difference between

$$n(n+p)(n+2p)\,(n+3p)\,\ldots\ldots\,(n+\overline{m+1}p)\,\ldots\;[1]$$

and $(n-p)\,(n)\,(n+p)\,(n+2p)\,\ldots\ldots\,(n+mp)\,\ldots\ldots\;[2]$

is $(m+2)\,pn\,(n+p)\,(n+2p)\,\ldots\,(n+mp)\,\ldots\ldots\;[3]$

and consequently that a series, of which the several terms are of the form

$$n(n+p)(n+2p)\dots(n+mp)$$

is equal to $\dfrac{1}{(m+2)p}$ the difference between two series whose terms are severally represented by [1] and [2].

1. For example, let n terms of the series

$$1.2+2.3+3.4+\dots\dots n(n+1)$$

be required: then the series [1], [2], are

$$\begin{array}{r} 1.2.3+2.3.4+3.4.5+\dots n(n+1)(n+2) \\ 0+1.2.3+2.3.4+\dots(n-1)n(n+1)\quad\quad\quad\ \\ \hline n(n+1)(n+2) \end{array}$$

And since $p=1$, $m=1$, $\therefore$ sum $=\dfrac{n(n+1)(n+2)}{3}$.

The proposed series is evidently the double of the series in example 5, at page 248, so that the number of shot in a triangular pile may be investigated in this way.

In this example the leading term of the second or subtractive series is evidently 0, there being always the same number of terms in each series. If the proposed series had commenced with 2 . 3, the subtractive series would have commenced with 1 . 2 . 3, the term immediately preceding that with which the upper series would then commence. And generally, in arranging the two series for subtraction, the lower series must always be extended towards the left, beyond the upper, to as many terms as the upper is extended beyond the lower, to the right; as is manifest, because there must always be the same number of terms in each series, the last term of the first being [1], and that of the second [2.]

It is further obvious that none but the terms in each series which thus project beyond those in the other, actually appear in the result of the subtraction adverted to: the intermediate terms, standing in pairs one under another, being equal, each to each, disappear from the remainder.

We might therefore easily frame a rule in words that should not require the writing down of these mutually opposing terms in the two series, but only those which project beyond them on each side, and which alone constitute the remainder sought: but, except in very easy examples, such as that given above and example 2 following, the verbal statement of the process would involve a complication that can never be felt in proceeding in every case conformably to the general algebraic principle above, as we have already done in the former parts of this chapter.

2. Let the series proposed for summation be

$$1.2.3 + 2.3.4 + 3.4.5 + \ldots n(n+1)(n+2)$$

where $m = 2$, $p = 1$.

Then from [1], [2], we have

$$1.2.3.4 + 2.3.4.5 + \ldots\ldots n(n+1)(n+2)(n+3)$$

$$0 + 1.2.3.4 + \ldots\ldots (n-1)(n)(n+1)(n+2)$$

$$\therefore \text{Sum} = \frac{n(n+1)(n+2)(n+3)}{4}.$$

3. Let the series be

$$1.4 + 2.5 + 3.6 + \ldots\ldots n(n+3)$$

where $p = 3$, $m = 1$.

$$1.4.7 + 2.5.8 + \ldots n(n+3)(n+6)$$

$$-2.1.4 - 1.2.5 + 0 + 1.4.7 + 2.5.8 + \ldots (n-3)(n)(n+3)$$

$$8 + 10 + (n-2)(n+1)(n+3) + (n-1)(n+2)(n+5) + n(n+3)(n+6),$$

therefore $\frac{1}{9}$th of this is the sum required.

From the close analogy between these operations and those so frequently exhibited in the former parts of this chapter—the same general principle pervading all—it seems unnecessary to extend these examples further. We shall merely add, that all series may be summed upon this principle, provided they admit of decomposition into component series, whose terms, like those above,

consist of factors in arithmetical progression: the following is an example; (see also p. 278:)

4. Required the sum of n terms of the series

$$1^2+2^2+3^2+4^2+\dots n^2.$$

Here $n^2=(n-1)n+n$, so that the series is composed of two others, whose final terms are respectively $(n-1)n$ and n. By the foregoing principle the sums of these latter are

$$\frac{(n-1)n(n+1)}{3} \text{ and } \frac{n(n+1)}{2}.$$

Hence, by adding these we have

$$\frac{n(n+1)(2n+1)}{6}$$

for the sum of the proposed series.

In this way, therefore, may we investigate the formula for the number of shot in a pyramidal pile whose base is a square.

(173.) For further and more interesting applications of the general principles established in the present chapter, the inquiring student may consult the *Mathematical Dissertations* by the author of the present treatise. The doctrine of infinite series is however a very comprehensive subject, and the general methods of summation, hitherto discovered, apply to comparatively but a very few classes. The known developments of algebra, as, for instance, those for a binomial, a logarithm, and an exponential, will often suggest the mode of proceeding in particular cases: we shall here give a few instances of this:

1. Required the sum of the infinite series,

$$x+x^2+x^3+x^4+\text{\&c.}$$

By the binomial theorem, or by common division,

$$(1-x)^{-1}=\frac{1}{1-x}=1+x+x^2+x^3+\text{\&c.}$$

$$\therefore \frac{x}{1-x}=x+x^2+x^3+x^4+\text{\&c.}$$

If $x = \frac{1}{2}$ then $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \&c. = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$

If $x = \frac{1}{3}$ then $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \&c. = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}.$

&c. &c.

2. Required the sum of the infinite series;

$$x - x^2 + x^3 - x^4 + \&c.$$

By the binomial theorem,

$$(1 + x)^{-1} = \frac{1}{1 + x} = 1 - x + x^2 - x^3 + \&c.$$

$$\therefore \frac{x}{1 + x} = x - x^2 + x^3 - x^4 + \&c.$$

If $x = \frac{1}{2}$ then $\frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \&c. = \frac{\frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{3}.$

If $x = 1$, then $1 - 1 + 1 - 1 + \&c. = \frac{1}{1 + 1} = \frac{1}{2}.$

In like manner, if the series had been

$$x \pm 2x^2 + 3x^3 \pm \&c.$$

we should have found for the sum the expression $\frac{x}{(1 \mp x)^2}.$

3. Required the sum S of the infinite series

$$x + 4x^2 + 9x^3 + 16x^4 + \&c.$$

By the binomial theorem,

$$\frac{1}{(1 - x)^3} = 1 + 3x + 6x^2 + 10x^3 + \&c.$$

$$\therefore \frac{x}{(1 - x)^3} = x + 3x^2 + 6x^3 + 10x^4 + \&c. \ldots [1]$$

Subtracting this from the proposed, we have

$$S - \frac{x}{(1 - x)^3} = x^2 + 3x^3 + 6x^4 + \&c.$$

And this by [1] is equal to $\frac{x^3}{1-x^3}$.

$$\therefore S = \frac{x(1+x)}{(1-x)^3}.$$

4. Required the sum of the infinite series,

$$\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \frac{x^4}{4.5} + \&c.$$

By article (139) the Napierian logarithm of $1-x$ is,

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \&c.\right)$$

$$\therefore \log\frac{1}{1-x} = -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \&c.$$

$$\therefore x\log\frac{1}{1-x} = \quad x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \&c.$$

and by subtracting,

$$(x-1)\log\frac{1}{1-x} = -x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} + \&c.$$

$\therefore$ transposing the x, and dividing by x,

$$1 + \frac{x-1}{x}\log\frac{1}{1-x} = \frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \&c.$$

5. Required the sum of the infinite series,

$$\frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} + \frac{x^7}{2.3.4.5.6.7} + \&c.$$

By the exponential theorem, (p. 210.)

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \&c.$$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{2.3} + \frac{x^4}{2.3.4} - \&c.$$

$$\therefore \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} + \&c.$$

$$\therefore \frac{e^x - e^{-x}}{2} - x = \text{sum}.$$

PROMISCUOUS EXAMPLES.

1. Required the sum of the series $\frac{3}{1\,.\,2\,.\,2}+\frac{4}{2\,.\,3\,.\,2^2}+\frac{5}{3\,.\,4\,.\,2^3}+$ &c. ad infinitum.

$$\left\{\begin{array}{l}\frac{3}{1\,.\,2}+\frac{4}{2\,.\,2^2}+\frac{5}{3\,.\,2^3}+\text{\&c.}\\ \quad-(\frac{3}{2\,.\,2}+\frac{4}{3\,.\,2^3}+\text{\&c.})\end{array}\right\}=\frac{3}{1\,.\,2}-(\frac{2}{2\,.\,2^2}+\frac{3}{3\,.\,2^3}+\text{\&c.})$$

$$=\frac{3}{1\,.\,2}-(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\text{\&c.})=\tfrac{3}{2}-\tfrac{1}{2}=1=\text{sum.}$$

2. Required the sum of the series $\frac{5}{1.2.3.2^2}+\frac{6}{2.3.4.2^3}+\frac{7}{3.4.5.2^4}+$ &c. ad infinitum.

Ans. $\frac{1}{4}$.

3. Required the sum of the series $x+3x^2+6x^3+10x^4+$ &c. ad infinitum.

Ans. $\frac{x}{(1-x)^3}$.

4. Required the sum of n terms of the series

$$\frac{1}{4\,.\,8}-\frac{1}{6\,.\,10}+\frac{1}{8\,.\,12}-\text{\&c.}$$

$$\text{Ans.}\left\{\begin{array}{l}\text{when } n \text{ is even, } \frac{n}{16\,(n+2)}-\frac{n}{24\,(n+3)}.\\ \text{when } n \text{ is odd, } \frac{n+4}{16\,(n+2)}-\frac{n+6}{24\,(n+3)}.\end{array}\right.$$

5. Required the sum of the series $\frac{1}{8\,.\,18}+\frac{1}{10\,.\,21}+\frac{1}{12\,.\,24}+\frac{1}{14\,.\,27}$ + &c. ad infinitum.

Ans. $\frac{3}{80}$.

6. Required the sum of the series $\frac{10\,.\,18}{2\,.\,4\,.\,9\,.\,12}+\frac{12\,.\,21}{4\,.\,6\,.\,12\,.\,15}+\frac{14\,.\,24}{6\,.\,8\,.\,15\,.\,18}+$ &c. ad infinitum.

Ans. $\frac{19}{24}$.

ON RECURRING SERIES.

(173.) A Recurring Series is one, each of whose terms, after a certain number, bears a uniform relation to the same number of those which immediately precede.

(174.) It is obvious that a variety of infinite series will arise from developing different fractional expressions; those, however, which generate recurring series are always of a particular form.

(175.) The fraction $\frac{a}{a' + b'x}$, for instance, is of this kind, for the series which arises from the actual division is recurring, thus:

$$
\begin{array}{l}
a' + b'x)\; a \quad \left(\frac{a}{a'} - \frac{ab'x}{a'^2} + \frac{ab'^2x^2}{a'^3} - \&c.\right. \\
\qquad\qquad a + \frac{ab'x}{a'} \\
\qquad\qquad \overline{\qquad\quad} \\
\qquad\qquad\quad - \frac{ab'x}{a'} \\
\qquad\qquad\quad - \frac{ab'x}{a'} - \frac{ab'^2x^2}{a'^2} \\
\qquad\qquad\quad \overline{\qquad\qquad\quad} \\
\qquad\qquad\qquad\qquad \frac{ab'^2x^2}{a'^2} \\
\qquad\qquad\qquad\qquad \frac{ab'^2x^2}{a'^2} + \frac{ab'^3x^3}{a'^3} \\
\qquad\qquad\qquad\qquad \overline{\qquad\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad - \frac{ab'^3x^3}{a'^3} \\
\qquad\qquad\qquad\qquad\qquad\quad \&c.
\end{array}
$$

where it is obvious that each term, commencing at the second,

is equal to that which immediately precedes, multiplied by $-\frac{b'x}{a'}$ which quantity is called the *scale of relation* of the terms, or $\frac{b'}{a'}$ is the scale of relation of the coefficients; therefore, representing the terms of the series by A, B, C, D, &c., we have

$$\text{A} = \frac{a}{a'}, \text{ whence } a'\text{A} - a = 0,$$

$$\text{B} = -\frac{b'x}{a'}\text{A} \ldots\ldots\ldots b'x\text{A} + a'\text{B} = 0,$$

$$\text{C} = -\frac{b'x}{a'}\text{B} \ldots\ldots\ldots b'x\text{B} + a'\text{C} = 0,$$

$$\text{D} = -\frac{b'x}{a'}\text{C} \ldots\ldots\ldots b'x\text{C} + a'\text{D} = 0,$$

&c. &c.

where we may observe that the coefficients of A, B; of B, C; of C, D, &c., are the terms of the denominator of the generating fraction taken in reverse order.

(176.) The fraction $\frac{a + bx}{a' + b'x + c'x^2}$ is another of this kind; for if this be developed like that above, and similar substitutions be made, there will be found to result

$$\text{A} = \frac{a}{a'}, \text{ whence } \ldots\ldots a'\text{A} - a \ldots\ldots\ldots\ldots = 0,$$

$$\text{B} = \frac{b - b'\text{A}}{a'} \ldots\ldots\ldots\ldots b'\text{A} + a'\text{B} - b \ldots\ldots = 0,$$

$$\text{C} = -\frac{c'x^2\text{A} + b'x\text{B}}{a'} \ldots\ldots c'x^2\text{A} + b'x\text{B} + a'\text{C} = 0,$$

$$\text{D} = -\frac{c'x^2\text{B} + b'x\text{C}}{a'} \ldots\ldots c'x^2\text{B} + b'x\text{C} + a'\text{D} = 0,$$

&c. &c.

where each term, commencing at the third, is equal to the two

immediately preceding multiplied respectively by $-\frac{c'x^2}{a'}, -\frac{b'x}{a'}$ which is, therefore, the scale of relation of the terms; also, the coefficients of A, B, C; of B, C, D, &c., are the terms of the denominator of the generating fraction, taken in reverse order.

(177.) The fraction $\frac{a + bx + cx^2}{a' + b'x + c'x^2 + d'x^3}$ is also one of the same kind, as its development will show, the scale of relation of the terms, in the resulting series, being $-\frac{d'x^3}{a'}, -\frac{c'x^2}{a'}, -\frac{b'x}{a'}$, commencing at the fourth term. And, in general, the development of any rational fraction of the form

$$\frac{a + bx + cx^2 + \dots . px^m}{a' + b'x + c'x^2 + \dots . q'x^{m+1}}$$

will be a recurring series, in which any term, commencing at the $m + 2$th, will be equal to the $m + 1$ preceding, multiplied by $-\frac{q'x^{m+1}}{a'}, -\frac{p'x^m}{a'}, \dots -\frac{c'x^2}{a'}, -\frac{b'x}{a'}$, respectively, which is, therefore, the scale of relation of the terms.

If $a' = 1$, then the scale of relation is $-q'x^{m+1}, -p'x^m, \dots . -c'x^2, -b'x$, being the several terms of the denominator taken in reverse order and with changed signs, the first term, 1, being omitted.

PROBLEM I.

To find the sum of an infinite recurring series.

Let A + B + C + D + ... + K + L + M + N represent a recurring series, and let it be supposed such, that each term, commencing at the fourth, depends upon the three preceding; then, as in Art. 176, we shall have, by supposing the terms in the denominator of the generating fraction to be p, q, r, s, the following equations, viz.

$$s\text{A} + r\text{B} + q\text{C} + p\text{D} = 0,$$

$$s\text{B} + r\text{C} + q\text{D} + p\text{E} = 0,$$

$$s\text{C} + r\text{D} + q\text{E} + p\text{F} = 0,$$

$$s\text{D} + r\text{E} + q\text{F} + p\text{G} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$s\text{K} + r\text{L} + q\text{M} + p\text{N} = 0,$$

and, taking the sum of these equations, we have

$$s(\text{A} + \text{B} + \text{C} + \text{D} + \ldots\ldots \text{K}) + r(\text{B} + \text{C} + \text{D} + \text{E} + \ldots\ldots \text{L})$$
$$+ q(\text{C} + \text{D} + \text{E} + \text{F} + \ldots\ldots \text{M}) + p(\text{D} + \text{E} + \text{F} + \text{G} + \ldots\ldots \text{N}) = 0;$$

which, by putting S for the sum, becomes the same as

$$s(\text{S} - \text{L} - \text{M} - \text{N}) + r(\text{S} - \text{A} - \text{M} - \text{N}) + q(\text{S} - \text{A} - \text{B} - \text{N})$$
$$+ p(\text{S} - \text{A} - \text{B} - \text{C}) = 0;$$

from which equation we get S =

$$\frac{p(\text{A} + \text{B} + \text{C}) + q(\text{A} + \text{B} + \text{N}) + r(\text{A} + \text{M} + \text{N}) + s(\text{L} + \text{M} + \text{N})}{p + q + r + s}$$

so that the sum may be determined from having the first three and last three terms with the scale of relation given: but if the series be infinite, and decreasing, the last three terms will vanish, and the sum will be

$$\frac{p(\text{A} + \text{B} + \text{C}) + q(\text{A} + \text{B}) + r\text{A}}{p + q + r + s} = \frac{\text{A}(p + q + r) + \text{B}(p + q) + \text{C}p}{p + q + r + s}$$

It is plain that a similar formula will be obtained, wherever the recurrence commences.

EXAMPLES.

1. Required the sum of the infinite recurring series

$$1 + 2x + 8x^2 + 28x^3 + 100x^4 + 356x^5 + \&c.$$

Here the scale of relation is $2x^2$, $3x$.

$\therefore$ the third term, $C = 2x^2A + 3xB$, whence

$$-2x^2A - 3xB + C = 0;$$

consequently, $s = -2x^2$, $r = -3x$, $q = 1$, and $p = 0$.

$$\therefore \text{sum} = \frac{A(1-3x) + B}{1 - 3x - 2x^2} = \frac{1 - x}{1 - 3x - 2x^2}.$$

2. Required the sum of the infinite recurring series

$$1 + 2x + 3x^2 + 5x^3 + 8x^4 + \&c.$$

the scale of relation being x^2, x.

Ans. $\dfrac{1+x}{1-x-x^2}$.

3. Required the sum of the infinite recurring series

$1 + 3x + 5x^2 + 7x^3 + \&c.$, the scale of relation being $-x^2$, $2x$.

Ans. $\dfrac{1+x}{(1-x)^2}$.

4. Required the sum of the infinite recurring series

$$3 + 5x + 7x^2 + 13x^3 + 23x^4 + \&c.,$$

the scale of relation being $-2x^3$, x^2, $2x$.

Ans. $\dfrac{3 - x - 6x^2}{1 - 2x - x^2 + 2x^3}$.

PROBLEM II.

To find the sum of any number of terms of a recurring series.*

This may be effected by means of the expression for S in the preceding problem, but more commodiously by subtracting from the sum of the series continued to infinity the sum of all those terms which follow the nth; thus, if the nth term of the recurring series A + B + C + &c. be T, then, putting S for the sum of all the terms to infinity, and S′ for the sum of those to infinity which follow T, we shall have, by last problem, $S - S' =$

$$\frac{A(p+q+r)+B(p+q)+Cp-U(p+q+r)-V(p+q)-Wp}{p+q+r+s}$$

$$=\frac{(A-U)(p+q+r)+(B-V)(p+q)+(C-W)p}{p+q+r+s}=$$

the sum of n terms.

---

* The finding the sum of a finite number of terms of a recurring series supposes that the general term of the series is previously known: to discover the general term is sometimes, however, the most perplexing part of the problem, it being often attended with considerable difficulties. The only direct way in which it can be discovered is derived from considering the generating fraction

$$\frac{a+bx+cx^2+\dots\dots px^m}{a'+b'x+c'x^2+\dots\dots q'x^{m+1}},$$

as the same as

$$(a+bx+cx^2+\dots\dots px^m)(a'+b'x+c'x^2+\dots\dots q'x^{m+1})^{-1},$$

which may be developed, and the general term of the resulting series obtained, by the MULTINOMIAL THEOREM. Another, though a less direct, method of discovering the general term when the series is of a high order, is by decomposing the generating fraction into others of simpler forms; but whichever method be employed, the operation in complicated series is one of much difficulty.

The student may on this subject consult LACROIX, *Complément d'Algèbré*, and *Elémens d'Algèbre* of BOURDON, in which works will be also found the method proposed by LAGRANGE for ascertaining whether or not a given series be recurring.

EXAMPLES.

1. Required the sum of n terms of the series

$$1 + 2x + 3x^2 + \ldots\ldots nx^{n-1}.$$

Here the scale of relation is $-x^2, 2x$;

$$\therefore \text{C} = -x^2\text{A} + 2x\text{B}, \text{ whence } x^2\text{A} - 2x\text{B} + \text{C} = 0,$$

$$\therefore s = x^2, r = -2x, q = 1, p = 0, \text{ also } \text{U} = (n+1)x^n,$$

$$\text{V} = (n+2)x^{n+1}, \text{ and } \text{W} = (n+3)x^{n+2};$$

$$\text{consequently, sum} = \frac{(\text{A} - \text{U})(1 - 2x) + \text{B} - \text{V}}{1 - 2x + x^2} =$$

$$\frac{[1 - (n+1)x^n](1 - 2x) + 2x - (n+2)x^{n+1}}{1 - 2x + x^2} =$$

$$\frac{1 - (n+1)x^n + nx^{n+1}}{1 - 2x + x^2} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2}.$$

2. Required the sum of n terms of the series

$$1 + 3x + 5x^2 + 7x^3 + \&c.$$

the scale of relation being $-x^2, 2x$.

Ans. $\dfrac{1 + x - (2n+1)x^n + (2n-1)x^{n+1}}{(1-x)^2}$.

ON THE METHOD OF INDETERMINATE COEFFICIENTS.

(178.) The method of Indeterminate Coefficients, which is used to develop algebraical expressions into series proceeding according to some uniform law, consists in assuming the proposed expression equal to a series of the desired form with indeterminate or unknown coefficients;* and if this assumed series be multiplied by the denominator of its equivalent fraction, or raised to the power necessary to free from radicals its equivalent surd, then, by equating the coefficients of the homologous terms in the resulting equation, the several values of the assumed coefficients will become known. If the assumed form of development be impossible, the conditional equations will either be contradictory, or else consistent only for infinite values of the assumed coefficients.

EXAMPLES.

1. Required the development of $\frac{a}{a'+b'x}$ by the method of indeterminate coefficients.

Assume $\frac{a}{a'+b'x} = \text{A} + \text{B}x + \text{C}x^2 + \text{D}x^3 + \text{\&c.}$

then, multiplying each side by $a'+b'x$, and transposing, we have

$$\left.\begin{matrix}\text{A}a' + \text{B}a' \\ -a + \text{A}b'\end{matrix}\right\} x \left.\begin{matrix}+\text{C}a' \\ +\text{B}b'\end{matrix}\right\} x^2 \left.\begin{matrix}+\text{D}a' \\ +\text{C}b'\end{matrix}\right\} x^3 + \text{\&c.} = 0\,;$$

---

* "Indeterminate" is used here in the sense of undetermined, or unknown; the term is certainly an objectionable one, though custom has sanctioned its use. In strictness, *determinate* is the proper word; the coefficients being determinate though unknown.

$$\text{whence}\left\{\begin{array}{ll} Aa' - a = 0, \text{ therefore } A = & \dfrac{a}{a'} \\ Ba' + Ab' = 0 \;\ldots\ldots\; B = & -\dfrac{b'}{a'}A \\ Ca' + Bb' = 0 \;\ldots\ldots\; C = & -\dfrac{b'}{a'}B \\ Da' + Cb' = 0 \;\ldots\ldots\; D = & -\dfrac{b'}{a'}C \\ \&c. & \&c. \end{array}\right.$$

$$\therefore \frac{a}{a'+b'x} = \frac{a}{a'} - \frac{b'}{a'}Ax - \frac{b'}{a'}Bx^2 - \frac{b'}{a'}Cx^3 - \&c.$$

the same as was before found from actual division at page 266.

2. Required the development of $\sqrt{a^2+x^2}$ by this method.

Assume $\sqrt{a^2+x^2} = A + Bx + Cx^2 + Dx^3 + \&c.$;

then, by squaring each side, and transposing, we have

$$\begin{array}{l} A^2 + 2ABx \\ -a^2 \end{array} \left.\begin{array}{r} +2AC \\ +B^2 \\ -1 \end{array}\right\} x^2 \left.\begin{array}{r} +2AD \\ +2BC \end{array}\right\} x^3 \left.\begin{array}{r} +2AE \\ 2BD \\ +C^2 \end{array}\right\} x^4 + \&c. = 0;$$

$$\text{whence}\left\{\begin{array}{ll} A^2 - a^2 = 0, \text{ therefore } & A = a \\ 2AB = 0 \;\ldots & B = 0 \\ 2AC - 1 = 0 \;\ldots & C = \dfrac{1}{2a} \\ 2AD + 2BC = 0 \;\ldots & D = 0 \\ \&c. & \&c. \end{array}\right.$$

$$\therefore \sqrt{a^2+x^2} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \&c.$$

3. Required the development of $\dfrac{x}{1+x+x^2}$ by the same method.

Here, since the first term of the series must contain x,

$$\text{assume } \frac{x}{1+x+x^2} = Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$$

then we have

$$\left.\begin{matrix}A\\-1\end{matrix}\right\} x \left.\begin{matrix}+A\\+B\end{matrix}\right\} x^2 \left.\begin{matrix}+A\\+B\\+C\end{matrix}\right\} x^3 \left.\begin{matrix}+B\\+C\\+D\end{matrix}\right\} x^4 \left.\begin{matrix}+C\\+D\\+E\end{matrix}\right\} x^5 + \&c. = 0;$$

$$\text{whence} \left\{\begin{array}{lll} A - 1 & = 0, \text{ therefore } & A = 1 \\ A + B & = 0 \quad . \quad . \quad . & B = -1 \\ A + B + C & = 0 \quad . \quad . \quad . & C = 0 \\ B + C + D & = 0 \quad . \quad . \quad . & D = 1 \\ C + D + E & = 0 \quad . \quad . \quad . & E = -1; \end{array}\right.$$

$$\therefore \frac{x}{1 + x + x^2} = x - x^2 + x^4 - x^5 + x^7 - \&c.$$

If x be taken equal to 1, we shall have

$$\frac{1}{3} = 1 - 1 + 1 - 1 + 1 - \&c. \; . \; . \; . \; [1]$$

But we have seen (162), that

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - \&c. \; . \; . \; . \; [2].$$

This apparent discrepancy is to be explained upon the same principles as those already discussed in the Scholium to the Binomial Theorem, page 203. In the right-hand members of the preceding equations, two really different things are implied under the same symbol; so that the contradiction is only apparent. The symbol adverted to is the &c., which has different values in the two expressions, and could not cancel one another in the subtraction of the upper series from the lower.

Moreover, in order to institute a comparison between the two series [1] and [2], the essential difference which exists between them, in their general forms, ought not to be obliterated in the special cases; preserving therefore the distinction—noting the deficient terms in [1] by zeros—the two series are

$$\frac{1}{3} = 1 - 1 + 0 + 1 - 1 + 0 + 1 - \&c. \; . \; . \; . \; [3]$$

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - \&c.$$

which cannot of course be regarded as identical, whatever number of terms in each be taken.

A great variety of other fractions, besides $\frac{x}{1+x}$ and $\frac{x}{1+x+x^2}$, generate series which, when $x=1$, give rise to the common form [1], when the essential distinction, observable in the general developments, is suppressed in this particular case.

The fraction $\frac{1+x}{1+x+x^2}$, for instance, furnishes the development

$$1-x^2+x^3-x^5+x^6-x^8+\&c.$$

which degenerates into the common form [1], like the others, when the distinguishing character of the series is obliterated; but if this be preserved, then, when $x=1$, we have

$$\frac{2}{3}=1+0-1+1+0-1+1+0-1+\&c. \ . \ . \ [4].$$

When no trace of the general form of the development is preserved, in the particular case in question, we cannot pass from it to the generating fraction; but when the character which marks the general form is presented to us in the particular case, the proper fraction to which that case belongs may always be determined.

For, by introducing the successive powers of x into the several terms of the series, we shall either get a geometrical series, or else a series that may easily be decomposed into geometrical series.

Thus if the series [3] be proposed, then, introducing the powers of x, we have

$$1-x+x^3-x^4+x^6-\&c. \ . \ . \ . \ [A],$$

which consists of the two geometrical series

$$1+x^3+x^6+x^9+\&c.,$$
$$\text{and } -x-x^4-x^7-x^{10}-\&c.,$$

the sums of which, to infinity, are

$$\frac{1}{1-x^3} \text{ and } -\frac{x}{1-x^3}.$$

Hence the sum of the series [A] is

$$\frac{1-x}{1-x^3}=\frac{1}{1+x+x^2},$$

which, when $x=1$ becomes $\frac{1}{3}$, the sum of the proposed series.

In like manner, for the sum of the series [4] we have, by supplying the powers of x,

$$1 - x^2 + x^3 - x^5 + x^6 - x^8 + \&c. =$$

$$\left.\begin{array}{l} 1 + x^3 + x^6 + x^9 + \&c. \\ -(x^2 + x^5 + x^8 + x^{11} + \&c.) \end{array}\right\} =$$

$$\frac{1}{1-x^3} - \frac{x^2}{1-x^3} = \frac{1-x^2}{1-x^3} = \frac{1+x}{1+x+x^2},$$

consequently, putting $x = 1$, the sum of the proposed series is $\frac{2}{3}$. And the same method may be employed in all other cases.

The result obtained in this way is always found to be the *average*, or *mean*, of all the results that would be furnished successively, in summing the proposed series, by stopping first at the first term, then at the second, then at the third, and so on up to the place at which the terms already gone over recur; by proceeding with the summation beyond this point, the results already obtained will merely arise over again. Thus, in summing the series [3], if we stop at the first term, we get 1; if at the second, 0; if at the third, 0; and no other results can be obtained, however far we continue the summation; the average or mean of these three results, is one third of their sum, that is $\frac{1}{3}$; and this is the sum of the proposed series.

Again, in summing the series [4], if we stop at the first term, we get 1; if at the second, 1; if at the third, 0; after which the same results recur: the average of them is, in this case, $\frac{2}{3}$; the sum of the series.

This method of finding the sums of series of this kind was first proposed by D. Bernoulli. (See Peacock's Report on Analysis, in vol. iii of the Proceedings of the British Association: also Note B at the end.)

4. Required the development of $\sqrt{1-x}$ by the method of indeterminate coefficients.

$$\text{Ans. } 1 - \frac{x}{2} - \frac{x^2}{2\,.\,4} - \frac{3x^3}{2\,.\,4\,.\,6} - \frac{3\,.\,5x^4}{2\,.\,4\,.\,6\,.\,8} - \&c.$$

5. Required the development of $\frac{1+2x}{1-x-x^2}$ by this method.

$$\text{Ans. } 1 + 3x + 4x^2 + 7x^3 + 11x^4 + \&c.$$

6. Required the development of $\frac{1}{1-2ax+x^2}$ by the above method.

$$\text{Ans. } 1 + 2ax + (4a^2 - 1)\,x^2 + (8a^3 - 4a)\,x^3 + \&c.$$

SCHOLIUM.

The foregoing applications of the method of indeterminate coefficients merely relate to the development of finite expressions in infinite series, or the decomposition of such expressions into their component monomials. But it is sometimes required to decompose an expression into other forms, the individual terms of which have not this ultimate degree of simplicity; and even to change an expression, consisting only of monomials, into another involving binomial or trinomial factors, &c.; thus in some measure reversing the procedure and objects of the foregoing article.

We have already remarked, in the chapter on Infinite Series, that in order to the successful application of the method of summation there delivered, it is requisite that the general term of the series consist of factors in arithmetical progression, (p. 261): or of the reciprocals of such; or else that it admit of being changed into a form into which only factors of this kind enter: an example of this change is given at page 262. We may now show that a like change may always be effected, by aid of the method of indeterminate coefficients, upon every rational expression of the form $ax^n + bx^{n-1} + cx^{n-2} + \&c.$: a single illustration will suffice for this. Let the expression proposed be $ax^2 + bx + c$, and assume

$$ax^2 + bx + c = Ax(x+1) + Bx + C = Ax^2 + (A+B)x + C,$$

$$\therefore a = A, \; B = b - a, \; C = c,$$

$$\therefore ax^2 + bx + c = ax(x+1) + (b-a)x + c.$$

Again. Let x^3 be proposed, and assume

$$x^3 = Ax(x+1)(x+2) + Bx(x+1) + Cx + D,$$

$$= Ax^3 + (3A+B)x^2 + (2A+B+C)x + D,$$

$$\therefore A = 1,\ 3A + B = 0,\ \therefore B = -3A = -3,\ 2A + B + C = 0,$$

$$\therefore C = -2A - B = 1, \text{ and } D = 0,$$

$$\therefore x^3 = x(x+1)(x+2) - 3x(x+1) + x.$$

For the decomposition of rational *fractions*, see Integral Calculus, Chapter II, and Herschel's Finite Differences.

ON THE MULTINOMIAL THEOREM.

(179.) The Multinomial Theorem is a formula which exhibits the general development of $(a + bx + cx^2 + dx^3 + \&c.)^{\frac{p}{q}}$ in a series ascending according to the powers of x. It may be investigated as follows:

Assume

$$(a + bx + cx^2 + \&c.)^{\frac{p}{q}} = A + Bx + Cx^2 + \&c.$$

Similarly,

$$(a + by + cy^2 + \&c.)^{\frac{p}{q}} = A + By + Cy^2 + \&c.$$

Put, for abridgment,

$$(a + bx + cx^2 + \&c.)^{\frac{1}{q}} = X,\ (a + by + cy^2 + \&c.)^{\frac{1}{q}} = Y;$$

then,

$$\frac{X^p - Y^p}{X^q - Y^q} = \frac{B(x-y) + C(x^2 - y^2) + D(x^3 - y^3) + \&c.}{b(x-y) + c(x^2 - y^2) + d(x^3 - y^3) + \&c.}$$

$$= \frac{B + C(x+y) + D(x^2 + xy + y^2) + \&c.}{b + c(x+y) + d(x^2 + xy + y^2) + \&c.}$$

Now when $x = y$ then $X = Y$, in which case we know (pa. 207) that he first side of the equation becomes

$$\frac{pX^{p-1}}{qX^{q-1}} = \frac{p}{q} \cdot X^{p-q} = \frac{p}{q}(a + bx + cx^2 + \&c.)^{\frac{p}{q}-1};$$

and the second side becomes

$$\frac{B + 2Cx + 3Dx^2 + \&c.}{b + 2cx + 3dx^2 + \&c.}$$

Multiplying, therefore, each of these sides by

$$(a + bx + cx^2 + \&c.)\ (b + 2cx + 3dx^2 + \&c.)$$

and we have

$$\frac{p}{q}\ (a + bx + cx^2 + \&c.)^{\frac{p}{q}}\ (b + 2cx + 3dx^2 + \&c.)$$

$$= (a + bx + cx^2 + \&c.)\ (B + 2Cx + 3Dx^2 + \&c.)$$

or substituting for simplicity's sake n for $\frac{p}{q}$, and putting the assumed series for the second factor in the first member of this equation, we have

$$n\ (A + Bx + Cx^2 + \&c.)\ (b + 2cx + 3dx^2 + \&c.) =$$
$$(a + bx + cx^2 + \&c.)\ (B + 2Cx + 3Dx^2 + \&c.)$$

that is, by actually performing the multiplications here indicated,

$$\left.\begin{array}{r} nAb + Bb \\ + 2Ac \end{array}\right| nx + \left.\begin{array}{r} Cb \\ + 2Bc \\ + 3Ad \end{array}\right| nx^2 + \left.\begin{array}{r} Db \\ + 2Cc \\ + 3Bd \\ + 4Ae \end{array}\right| nx^3 + \&c.$$

is equal to

$$\left.\begin{array}{r} Ba + 2Ca \\ + Bb \end{array}\right| x + \left.\begin{array}{r} 3Da \\ + 2Cb \\ + Bc \end{array}\right| x^2 + \left.\begin{array}{r} 4Ea \\ + 3Db \\ + 2Cc \\ + Bd \end{array}\right| x^3 + \&c.$$

Now A is obviously $= a^n$, therefore, by comparing the coefficients of the like terms in the above expressions, we shall have

$$\text{B}a = n\text{A}b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{whence } \text{B} = na^{n-1}b,$$

$$2\text{C}a = (n-1)\,\text{B}b + 2n\text{A}c \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{C} = \frac{(n-1)\,\text{B}b + 2na^nc}{2a},$$

$$3\text{D}a = (n-2)\,\text{C}b + (2n-1)\,\text{B}c + 3n\text{A}d \quad . \quad . \quad . \quad \text{D} = \frac{(n-2)\,\text{C}b + (2n-1)\,\text{B}c + 3na^nd}{3a},$$

$$4\text{E}a = (n-3)\,\text{D}b + (2n-2)\,\text{C}c + (3n-1)\,\text{B}d + 4n\text{A}e \quad . \quad \text{E} = \frac{(n-3)\,\text{D}b + (2n-2)\,\text{C}c + (3n-1)\,\text{B}d + 4na^ne}{4a},$$

&c. &c. &c.

Hence it appears that

$$(a + bx + cx^2 + \&c.)^n = a^n + na^{n-1}bx + \frac{(n-1)\,\text{B}b + 2na^nc}{2a}x^2 + \frac{(n-2)\,\text{C}b + (2n-1)\,\text{B}c + 3na^nd}{3a}x^3 + \frac{(n-3)\,\text{D}b + (2n-2)\,\text{C}c + (3n-1)\,\text{B}d + 4na^ne}{4a}x^4 + \&c.$$

where B represents the coefficient of the second term, C that of the third, D that of the fourth, &c. Or, if we put $na^{n-1} = \text{Q}$, this equation will become

$$(a + bx + cx^2 + \&c.)^n = a^n + \text{Q}bx + \left(\frac{(n-1)\,\text{B}b}{2a} + \text{Q}c\right)x^2 + \left(\frac{(n-2)\,\text{C}b + (2n-1)\,\text{B}c}{3a} + \text{Q}d\right)x^3 + \left(\frac{(n-3)\,\text{D}b + (2n-2)\,\text{C}c + (3n-1)\,\text{B}d}{4a} + \text{Q}e\right)x^4 + \&c.$$

which is a very commodious form for practice.

In the application of this theorem to particular examples, it is to be observed that whenever any of the powers of x are wanting they must be introduced, with zero coefficients, before the theorem is employed.

EXAMPLES.

1. What is the cube of the series $1 + x + x^2 + x^3 + x^4 +$ &c.?

Here $n = 3$, and a, b, c, &c. are each $= 1$, also $Q = 3$, therefore

$$a^n = 1 = A,$$

$$Qb = 3 = B,$$

$$\frac{(n-1)\,Bb}{2a} + Qc = 6 = C,$$

$$\frac{(n-2)\,Cb + (2n-1)\,Bc}{3a} + Qd = 10 = D,$$

$$\frac{(n-3)\,Db + (2n-2)\,Cc + (3n-1)\,Bd}{4a} + Qe = 15 = E,$$

&c.

$$\therefore (1 + x + x^2 + x^3 + \&c.)^3 = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \&c.$$

2. What is the square root of the series $1 + x + x^2 + x^3 +$ &c.?

Here $n = \frac{1}{2}$, and a, b, c, &c. are each $= 1$, also $Q = \frac{1}{2}$, therefore

$$a^n = 1 = A,$$

$$Qb = \tfrac{1}{2} = B,$$

$$\frac{(n-1)\,Bb}{2a} + Qc = \tfrac{3}{8} = C,$$

$$\frac{(n-2)\,Cb + (2n-1)\,Bc}{3a} + Qd = \tfrac{5}{16} = D,$$

$$\frac{(n-3)\,Db + (2n-2)\,Cc + (3n-1)\,Bd}{4a} + Qe = \tfrac{35}{128} = E;$$

&c.

$$\therefore (1 + x + x^2 + x^3 + \&c.)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \tfrac{5}{16}x^3 + \tfrac{35}{128}x^4 + \&c.$$

3. What is the cube of the series $2x + 3x^2 + 4x^3 +$ &c.?

Ans. $8x^3 + 36x^4 + 102x^5 + 231x^6 +$ &c.

4. What is the square of the series $1 - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 +$ &c.? or, which is the same thing, of $1 + 0x + 0x^2 - \frac{1}{3}x^3 + 0x^4 + \frac{1}{5}x^5 + 0x^6 - \frac{1}{7}x^7$ + &c.?

Ans. $1 - \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{1}{9}x^6 - \frac{2}{7}x^7 - \frac{2}{15}x^8 +$ &c.

5. What is the cube root of the series $1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 +$ &c.?

Ans. $1 + \frac{1}{6}x + \frac{1}{12}x^2 + \frac{35}{648}x^3 + \frac{383}{9720}x^4$ &c.

ON THE REVERSION OF SERIES.

(180.) To revert a series is to express the value of the unknown quantity in it by means of another series involving the powers of the quantity to which the proposed series is equal.

1. Let the series be of the form $ax + bx^2 + cx^3 +$ &c. $= y$; then, in order to express the value of x in terms of y, assume $x = Ay + By^2 + Cy^3$ + &c., and substitute this value for x in the proposed series, which will, in consequence, become, when y is transposed,

$$\left.\begin{matrix} aA \\ -1 \end{matrix}\right\} y \left.\begin{matrix} +aB \\ +bA^2 \end{matrix}\right\} y^2 \left.\begin{matrix} +aC \\ +2bAB \\ +cA^3 \end{matrix}\right\} y^3 \left.\begin{matrix} +aD \\ +2bAC \\ +bB^2 \\ +3cA^2B \\ +dA^4 \end{matrix}\right\} y^4 + \&c. = 0,$$

$$\text{whence} \begin{cases} aA - 1 \quad \ldots\ldots\ldots \quad = 0, \therefore A = \dfrac{1}{a}, \\ aB + bA^2 \quad \ldots\ldots\ldots \quad = 0 \ \ldots \ B = -\dfrac{b}{a^3}, \\ aC + 2bAB + cA^3 \quad \ldots\ldots \quad = 0 \ \ldots \ C = \dfrac{2b^2 - ac}{a^5}, \\ aD + 2bAC + bB^2 + 3cA^2B + dA^4 = 0 \ \ldots \ D = -\dfrac{5b^3 - 5abc + a^2d}{a^7}, \\ \&c. \qquad\qquad\qquad\qquad \&c. \end{cases}$$

$$\text{consequently, } x = \frac{1}{a}y - \frac{b}{a^3}y^2 + \frac{2b - ac}{a^5}y^3 - \frac{5b^3 - 5abc + a^2d}{a^7}y^4 + \&c.$$

2. If the series be of the form $ax + bx^3 + cx^5 +$ &c., where the even powers of x are absent, then we shall have, instead of the above,

$$x = \frac{1}{a}y - \frac{b}{a^4}y^3 + \frac{3b^2 - ac}{a^7}y^5 - \frac{12b^3 + a^2d - 8abc}{a^{10}}y^7 + \text{\&c.,*}$$

so that if the even powers are absent from the proposed series they are also absent from the reverted series.

EXAMPLES.

1. Given the series $x + x^2 + x^3 + \text{\&c.} = y$, to express the value of x in terms of y.

Here a, b, c, &c. are each 1;

$$\text{therefore } \frac{1}{a} = 1,$$

$$-\frac{b}{a^3} = -1,$$

$$\frac{2b^2 - ac}{a^5} = 1,$$

$$-\frac{5b^3 - 5abc + a^2d}{a^7} = -1,$$

&c. &c.

$$\therefore x = y - y^2 + y^3 - y^4 + \text{\&c.}$$

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* When the series is expressed by means of another, as

$$ax + bx^2 + cx^3 + \text{\&c.} = \alpha y + \beta y^2 + \gamma y^3 + \text{\&c.}$$

the value of x is to be obtained exactly in the same way, by assuming $x = \text{A}y + \text{B}y^2 + \text{C}y^3 + \text{\&c.}$, and substituting this value in the place of x in the first series, as above.

2. It is required to revert the series

$$2x+3x^3+4x^5+5x^7+\&c.=y.$$

Here $a=2$, $b=3$, $c=4$, &c.

$$\text{therefore } \frac{1}{a}=\tfrac{1}{2},$$

$$-\frac{b}{a^4}=-\tfrac{3}{16},$$

$$\frac{3b^2-ac}{a^7}=\tfrac{19}{128},$$

$$-\frac{12b^3+a^2d-8abc}{a^{10}}=-\tfrac{152}{1024};$$

$$\therefore x=\tfrac{1}{2}y-\tfrac{3}{16}y^3+\tfrac{19}{128}y^5-\tfrac{152}{1024}y^7+\&c.$$

3. Given the series $x-\frac{1}{2}x^2+\frac{1}{4}x^3-\frac{1}{8}x^4+\&c.=y$, to find the value of x in terms of y.

Ans. $x=y+\frac{1}{2}y^2+\frac{1}{4}y^3+\frac{1}{8}y^4+\&c.$

4. Given the series $x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\&c.=y$, to find the value of x in terms of y.

Ans. $x=y+\frac{1}{3}y^3+\frac{2}{15}y^5+\frac{17}{315}y^7+\&c.$

5. Given the series $1+x+\dfrac{x^2}{2}+\dfrac{x^3}{2\,.\,3}+\dfrac{x^4}{2\,.\,3\,.\,4}+\&c.=y$, to find the value of x in terms of y.

Ans. $x=y-1-\dfrac{(y-1)^2}{2}+\dfrac{(y-1)^3}{3}-\&c.$*

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* We know, from what has been said of logarithms, that this value of x is the Napierian logarithm of y; but if $x=\log y$, $\therefore e^x=y$, consequently,

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2\,.\,3}+\&c.,$$

which is the exponential theorem otherwise established at page 210.

CHAPTER IX.

ON INDETERMINATE EQUATIONS.

(181.) EQUATIONS are said to be indeterminate or unlimited when they admit of an indefinite or unlimited number of solutions, which they will always do when the number of unknown quantities exceeds the number of independent equations. The equation $ax - by = c$, for instance, is unlimited; for $x = \frac{c + by}{a}$, where y may be any value whatever; therefore x and y admit of an infinite number of values that will satisfy the equation $ax - by = c$; and such must evidently be always the case when one of the unknowns is expressible only by means of another unknown, each then admitting of an infinite number of values. The number of solutions in *integer* numbers is, however, often determinable. If, for instance, $ax + by = c$, then $x = \frac{c - by}{a}$; and, therefore, to have integer values of x and y, the question will be limited to the finding all the integer values of y that will make $\frac{c - by}{a}$ an integer. The limits of possibility, in equations of this kind, will be investigated in the following propositions.

Def. A *prime* number is a number that cannot be decomposed into factors; the only divisors of it being the number itself and unity. Such are the numbers 3, 5, 7, 11, 13, &c.

Two or more numbers are said to be *prime to each other*, when they do not admit of a common divisor: thus, 7, 8, and 9 are all prime to each other.

PROPOSITION I.

If a and b be any two numbers prime to each other, and if each of the terms

$$b, 2b, 3b, 4b, \quad . \quad . \quad . \quad . \quad (a-1)b,$$

be divided by a, all the resulting remainders will be different.

For, if it be supposed that the remainders will not all be different, let any two of the above terms, as mb, nb, leave the same remainder r; then, representing the respective quotients by q, q', we must have

$$qa + r = mb,$$

$$\text{and } q'a + r = nb;$$

therefore, by subtraction, $a(q-q') = b(m-n)$, whence $\frac{b(m-n)}{a}$ is an integer number; but neither b nor $m-n$ is divisible by a, the former being prime to it, and the latter less than a, since both m and n are less; therefore $\frac{b(m-n)}{a}$ cannot be an integer, and, consequently, the supposition cannot be admitted.

Cor. 1. Hence, since the remainders are all different, and are $a-1$ in number, each being necessarily less than a, it follows that they include all numbers from 1 to $a-1$.

Cor. 2. Therefore, since some one of the remainders will be 1, it follows that some number x less than a may be found that will make $bx-1$ exactly divisible by a; or, which is the same thing, the equation $bx-ay=1$ is always possible in positive integers, if a and b be prime to each other.

If, however, a and b be not prime to each other, the equation will be impossible in integers; for a and b having, in this case, a common measure, one side of the equation $bx-ay=1$, would be divisible by it, and the other not.

Cor. 3. Since $bx-ay=1$ is always possible, it follows, by changing the signs, that $ay-bx=-1$ is also possible; hence $ax-by=\pm 1$ is always possible in positive integers, if a and b be prime to each other.

PROPOSITION II.

If a and b be prime to each other, the equation

$$ax - by = \pm c$$

will admit of an infinite number of solutions in positive integers.

For, since the equation $ax' - by' = \pm 1$ is possible, the equation

$$acx' - bcy' = \pm c$$

is possible, which, by putting x for cx', and y for cy', becomes

$$ax - by = \pm c,$$

being the same as the proposed equation.

Let now one solution be $x = p$, and $y = q$, then

$$ap - bq = ax - by, \text{ or } ax - ap = by - bq,$$

$$\therefore \frac{a(x-p)}{b(y-q)} = 1, \text{ and therefore } \frac{x-p}{y-q} = \frac{b}{a} = \frac{mb}{ma},$$

$$\text{or } x - p = mb, \text{ and } y - q = ma;$$

$$\therefore x = p + mb, \text{ and } y = q + ma;$$

and since m may be any value whatever, from 0 to infinity, the number of positive values of x and y will be infinite.

Cor. Since p and q are integers, and since m may be either positive or negative, m may be so assumed, that x shall be less than b, or that y shall be less than a: for making m equal to 0, -1, -2, -3, &c. successively, we shall have

$$x = p, p - b, p - 2b, \text{ \&c. successively,}$$

$$\text{and } y = q, q - a, q - 2a, \text{ \&c. successively,}$$

where it is obvious that one of the values of x must be less than b, and one of the values of y less than a, whatever be the values of p and q.

It is plain that if a and b have a common measure which is not also a measure of c, the solution will be impossible in integers: for, dividing by this common measure, the second member of the equation would be a fraction, and the first (if integral values of x and y could exist) would be an integer, which is absurd.

PROPOSITION III.

The equation $ax+by=c$ is always possible in positive integers, if a and b be prime to each other, and if

$$c > (ab-a-b).*$$

For let $c=(ab-a-b)+r$, then the equation becomes

$$ax+by=(ab-a-b)+r,$$

which is possible if

$$x=\frac{ab-a-b-by+r}{a}=b-1-\frac{(y+1)b-r}{a}$$

be an integer; but, since $b-1$ is an integer, the possibility depends upon

$$\frac{(y+1)b-r}{a}=x'$$

being an integer not exceeding $b-1$; or, putting $y+1=y'$, upon the possibility of the equation $by'-ax'=r$, which has been already established (Prop. II); let then y' be less than a, or $y+1<a$ (Prop. II, Cor.), then, in the equation

$$\frac{(y+1)b-r}{a}=x',$$

x' must be less than b, and it therefore follows that

$$x=b-1-\frac{(y+1)b-r}{a}=b-1-x'$$

must be either zero or else some positive integer number; hence the equation $ax+by=c$, is always possible when a and b are prime to each other, and $c>(ab-a-b)$.

* It is to be understood that *zero* is always included among integer numbers, unless specially provided against. The only positive integral solution of $3x+2y=4$ is $x=0$, $y=2$.

SCHOL. The last two propositions will evidently be useful in discovering the possibility or impossibility of equations of this kind, and also in enabling us to propose them with proper restrictions. But with respect to this Prop. III, it must be observed that the equation is not necessarily *impossible*, though the above inequality have *not* place: in $10x + 3y = 16$ for instance, $x = 1$, $y = 2$.

PROBLEM I.

To find the integer values of x and y in the equation

$$ax \pm by = c.$$

Since $x = \frac{by + c}{a}$ must be a whole number, it follows that if the division of $by + c$ by a be actually performed, that the remainder $py + d$ must be divisible by a, that is, $\frac{py + d}{a}$ must represent a whole number; also, if the difference between $\frac{ay}{a}$ and the nearest multiple to it of $\frac{py + d}{a}$ be taken, the remainder, which may be represented by $\frac{qy + e}{a}$, must be a whole number, and q must be less than p; if again the difference of $\frac{py + d}{a}$, and the nearest multiple to it of $\frac{qy + e}{a}$ be taken, the remainder, which may be represented by $\frac{ry + f}{a}$, must also be a whole number, and r will be less than q; hence, by proceeding in this way, we shall at length arrive at a remainder of the form $\frac{y + k}{a}$, in which the coefficient of y is 1. Now the least positive value that can be given to y, in order that this expression may be a whole number, will evidently, when k is negative, be equal to the remainder arising from the division of k by a; but, when k is positive, the least value will be equal to a, *minus* this remainder. Hence, since the subtraction of fractions does not produce any change on the common denominators, the numerators only being operated upon, the process will be the same in effect by the following rule.

(182.) Having reduced the given equation to the form $x = \frac{by + c}{a}$, perform the division of $by + c$ by a, and call the remainder $py + d$. Take the difference of ay and the nearest multiple to it of $py + d$; then the difference of $py + d$ and nearest multiple to it of the remainder; then the difference of the preceding remainder and the nearest multiple to it of this last, and so on, till we get a remainder of the form $y - k$, or $y + k$, when the least value of y will, in the former case, be the remainder R, arising from dividing k by a, and in the latter case it will be a *minus* R.*

If the coefficients a, b of the proposed equation have a common measure which is not at the same time a measure of c, the equation will be impossible in integers (Prop. II). The result arrived at by the application of the preceding rule will always sufficiently indicate this circumstance, without our first inquiring whether a and b are prime to each other—or, being prime to each other, whether the condition of Prop. III has place. In the first case the final coefficient of y will be 0, and in the second the greatest value of x will be *negative*, if positive values are impossible.

EXAMPLES.

1. Given $21x + 17y = 2000$, to find all the positive values of x and y in whole numbers.

Here $x = \frac{2000 - 17y}{21} = 95 - \frac{17y - 5}{21}$, and $a = 21$,

$$\begin{array}{l} 21y \qquad = ay \\ 17y - 5 = py - d \\ \hline 4y + 5 \\ 4 \\ \hline 16y + 20 \\ \hline y - 25 = y - k; \\ \hline \end{array}$$

* This rule does not differ much from one given by Mr. Nicholson, in the 'Mathematical Companion' for 1819; it appears, however, to be rather more simple.

now $\frac{25}{21}$ gives a remainder $= 4 =$ the least value of y, which, substituted in the above expression for x, gives $\frac{2000-68}{21} = 92 =$ the greatest value of x; and by adding 21, the coefficient of x, continually to the least value of y, and subtracting 17, the coefficient of y, from the greatest value of x, we shall have all the possible values as follow:

$x = 92$	75	58	41	24	7
$y = 4$	25	46	67	88	109.

2. Given $19x = 14y - 11$, to find x and y in positive whole numbers.

Here $x = \frac{14y-11}{19}$, $y = \frac{19x+11}{14} = x + \frac{5x+11}{14}$, and $a = 14$.

$$\begin{array}{r} 5x + 11 \\ 3 \\ \hline 15x + 33 \\ 14x \\ \hline x + 33 \\ \hline \end{array}$$

Now $\frac{33}{14}$ gives a remainder $= 5$, $\therefore 14 - 5 = 9 =$ the least value of x; and, since in this example the less x is, the less y will be, we have, by substitution, $\frac{171+11}{14} = 13 =$ the least value of y, the number of solutions being indefinite.

3. Exhibit the number of different ways in which it is possible to pay 20*l.* in half-guineas and half-crowns only.

Let x represent the half-guineas, and y the half-crowns; then, by reducing to sixpences, we have

$$21x + 5y = 800;$$

$$\therefore x = \frac{800 - 5y}{21} = 38 - \frac{5y-2}{21},$$

$$\begin{array}{r} 5y - 2 \\ 4 \\ \hline 20y - 8 \\ 21y \\ \hline y + 8 \\ \hline \end{array}$$

$\therefore$ R $= 8$, and $21 - 8 = 13 =$ the least value of y; and $\therefore$ 35 is the greatest value of x; consequently, if we add 21 continually to the least value of y, and subtract 5 from the greatest value of x, we shall have all the possible values; thus,

$x = 35$	30	25	20	15	10	5
$y = 13$	34	55	76	97	118	139

or the number of solutions, besides the one first obtained, might have been determined without this trouble; for the number of times that 5 can be continually subtracted from 35, so that the remainders may be all positive, is evidently one less than the quotient of 35 by 5, viz. 6: had this division left a remainder, the number of solutions would have been a unit more, that is, the whole quotient.

4. Given $5x + 11y = 254$, to find all the different values of x and y in positive whole numbers.

Ans. $x =$	9	20	31	42
$y =$	19	14	9	4

5. Given $11x + 35y = 500$, to find the least integer value of x.

Ans. 20.

6. Given $19x - 117y = 11$, to find the least integral values of x and y.

Ans. $x = 56$, and $y = 9$.

7. Is it possible to pay 50*l.* by means of guineas and three-shilling pieces only?

Ans. Impossible.

8. A person bought sheep and lambs for 8 guineas, the sheep cost 1*l.* 6*s.* a piece, and the lambs 15*s.* How many of each did he buy?

Ans. 3 sheep and 6 lambs.

9. Is the equation $7x + 13y = 71$ possible or impossible in whole numbers?

Ans. Impossible.

PROBLEM II.

To determine *à priori* the number of solutions that the equation

$$ax + by = c$$

will admit of in positive integers, zero being excluded.

Let such integral values of x' and y' be found, that we may have $ax' - by' = 1$, which we have shown to be always possible (Prop. 1, Cor. 2);

$$\text{then, } acx' - bcy' = c, \therefore ax + by = acx' - bcy',$$

and, consequently, we may put $x = cx' - mb$, and $y = mq - cy'$, where m may be any number taken at pleasure, that will make these values of x and y positive integers; but, if no such value of m can be found, it will be a proof that the proposed equation is impossible in positive integers; and, on the contrary, as many suitable values of m as can be found, so many solutions will the equation admit of, and no more. Hence, because we must have $cx' > mb$, and $cy' < ma$, the whole number of solutions will be expressed by the difference between the integral parts of

$$\frac{cx'}{b} \text{ and } \frac{cy'}{a};$$

because, as m must be less than the first of these fractions, and greater than the second, the difference of their integral parts will evidently express the number of different values of m, except when $\frac{cx'}{b}$ is a complete integer; in which case, since $m < \frac{cx'}{b}$, the difference of the integral parts would be one more than the number of different values of m, therefore, when the expression $\frac{cx'}{b}$ is an integer, we must consider $\frac{b}{b}$ as a fraction, and reject it therefrom; but this must not be done with the other quantity $\frac{cy'}{a}$, because $m > \frac{cy'}{a}$.

EXAMPLES.

1. Required the number of solutions that the equation $9x + 13y = 2000$ will admit of in positive integers.

In the equation $9x' - 13y' = 1$, we have

$$x' = \frac{13y' + 1}{9} = y' + \frac{4y' + 1}{9};$$

therefore,

$$\begin{array}{l} 4y' + 1 \\ 2 \\ \hline 8y' + 2 \\ 9y' \\ \hline y' - 2 \\ \hline \end{array}$$

$\therefore y' = 2$, and $x' = 3$, hence the number of solutions is the integral part of $\frac{2000 \times 3}{13}$ *minus* the integral part of $\frac{2000 \times 2}{9}$, which is 17.

2. In how many different ways is it possible to pay 140*l.* by means of guineas and three-shilling pieces only?

Ans. The payment is impossible.

3. In how many different ways can 1000*l.* be paid in crowns and guineas? Ans. 190.

PROBLEM III.

To find the positive integer values of x, y, z, in the equation

$$ax + by + cz = d.$$

Let c be the greatest coefficient in this equation, then, since the values of x and y cannot be less than 1, the value of z cannot be greater than

$$\frac{d - a - b}{c};$$

if, therefore, we ascertain this limit, and then proceed as in Prob. I, we shall at length arrive at a remainder of the form $y \pm z \pm k$, where, if 1, 2, 3, &c. up to the limit, be successively substituted for z, all the values of x and y may be exhibited, as in Prob. I.

EXAMPLES.

1. Given $3x + 5y + 7z = 100$, to exhibit all the different values of x, y, and z, in positive integers, zero being excluded.*

Here z cannot be greater than $\dfrac{100 - 3 - 5}{7} = 13\frac{1}{7}$;

and by proceeding as in Prob. I,

$$x = \frac{100 - 5y - 7z}{3} = 33 - y - 2z - \frac{2y + z - 1}{3};$$

$$\begin{array}{r} 3y \\ 2y + z - 1 \\ \hline y - z + 1 \\ \hline \end{array}$$

now, by taking $z = 1$, y becomes $= 0$, and $x = 31$; but this answer is inadmissible, because $y = 0$ is to be excluded; but, by adding 3, the coefficient of x, to this value of y, and subtracting 5, the coefficient of y, from the value of x, we shall obtain another answer; and, by repeating this process continually, we shall obtain all the possible values of x and y, for this value of z; and in a similar manner are the values

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* This example is the same as that given by Mr. Bonnycastle, at page 232, vol. i, of his Algebra, where he finds the number of solutions to be 7, "which," he inadvertently says, "are all the integer values of x, y, z, that can be obtained from the given equation:" from the solution here given, however, it appears that 41 is the whole number of solutions.

of x and y to be found when $z = 2$, &c., when all the possible solutions will be found to be 41 in number, and to be as follow:

$$z = 1\left\{\begin{array}{l|c|c|c|c|c|} y = 3 & 6 & 9 & 12 & 15 & 18 \\ x = 26 & 21 & 16 & 11 & 6 & 1 \end{array}\right.$$

$$z = 2\left\{\begin{array}{l|c|c|c|c|c|} y = 1 & 4 & 7 & 10 & 13 & 16 \\ x = 27 & 22 & 17 & 12 & 7 & 2 \end{array}\right.$$

$$z = 3\left\{\begin{array}{l|c|c|c|c|} y = 2 & 5 & 8 & 11 & 14 \\ x = 23 & 18 & 13 & 8 & 3 \end{array}\right.$$

$$z = 4\left\{\begin{array}{l|c|c|c|} y = 3 & 6 & 9 & 12 \\ x = 19 & 14 & 9 & 4 \end{array}\right.$$

$$z = 5\left\{\begin{array}{l|c|c|c|} y = 1 & 4 & 7 & 10 \\ x = 20 & 15 & 10 & 5 \end{array}\right.$$

$$z = 6\left\{\begin{array}{l|c|c|c|} y = 2 & 5 & 8 & 11 \\ x = 16 & 11 & 6 & 1 \end{array}\right.$$

$$z = 7\left\{\begin{array}{l|c|c|} y = 3 & 6 & 9 \\ x = 12 & 7 & 2 \end{array}\right.$$

$$z = 8\left\{\begin{array}{l|c|c|} y = 1 & 4 & 7 \\ x = 13 & 8 & 3 \end{array}\right.$$

$$z = 9\left\{\begin{array}{l|c|} y = 2 & 5 \\ x = 9 & 4 \end{array}\right.$$

$$z = 10\left\{\begin{array}{l|} y = 3 \\ x = 5 \end{array}\right.$$

$$z = 11\left\{\begin{array}{l|c|} y = 1 & 4 \\ x = 6 & 1 \end{array}\right.$$

$$z = 12\left\{\begin{array}{l|} y = 2 \\ x = 2 \end{array}\right.$$

It is obvious, from the above, that when the solutions are very numerous, the process will become tedious; but there is seldom any necessity to exhibit all the solutions at length, as is done here, since the object of inquiry is not so much to find the solutions themselves, as to determine, *à priori*, the number that the equation admits of; the method of doing which will be pointed out in the next Problem: we shall, therefore, merely add another example to this problem, as an exercise for the student.

2. Given $17x + 19y + 21z = 400$, to exhibit all the different values of x, y, and z, in positive integers.

$$\text{Ans.}\left\{\begin{array}{l|c|c|c|c|c|c|c|c|c|} z = 1 & 2 & 3 & 4 & 5 & 6 & 11 & 12 & 13 & 14 \\ y = 11 & 9 & 7 & 5 & 3 & 1 & 8 & 6 & 4 & 2 \\ x = 10 & 11 & 12 & 13 & 14 & 15 & 1 & 2 & 3 & 4 \end{array}\right.$$

PROBLEM IV.

To determine the number of solutions that the equation

$$ax + by + cz = d$$

will admit of, two, at least, of the coefficients a, b, c, being prime to each other,* and zero values being excluded.

By Prob. II, the number of solutions that the equation $ax + by = c$ will admit of, is expressed by the integral parts of

$$\frac{cx'}{b} - \frac{cy'}{a},$$

x' and y' being determined from the equation $ax' - by' = 1$;

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* When this is not the case, the proposed equation must be transformed to another, that shall have two, at least, of its coefficients prime to each other. Thus, if the equation be

$$12x + 15y + 20z = 100001,$$

by transposing $20z$, and dividing by 3, we have

$$4x + 5y = 33334 - 7z + \frac{z-1}{3};$$

$\therefore \frac{z-1}{3}$ is an integer, which call u, then $z = 3u + 1$; whence, by substitution, the proposed equation becomes

$$12x + 15y + 20(3u + 1) = 100001,$$

which, by transposing the 20, becomes divisible by 3, and we then have

$$4x + 5y + 20u = 33327;$$

in which equation x and y have, of course, the same values as in the one proposed, and therefore the number of solutions must be the same; but it must be remembered that, in this last, one value of u may be 0, because

$$z = 3u + 1.$$

therefore, in the equation $ax + by = d - cz$, if we make $z = 1, 2, 3, 4$, &c. successively, then the number of solutions

$$\text{in the equation}\left\{\begin{array}{ll} ax+by=d-c \text{ will be the integ parts of} & \dfrac{(d-c)x'}{b} - \dfrac{(d-c)y'}{a} \\[2ex] ax+by=d-2c \quad \ldots\ldots\ldots & \dfrac{(d-2c)x'}{b} - \dfrac{(d-2c)y'}{a} \\[2ex] ax+by=d-3c \quad \ldots\ldots\ldots & \dfrac{(d-3c)x'}{b} - \dfrac{(d-3c)y'}{a} \end{array}\right.$$

&c. &c. &c.

the sum of which will be the whole number of solutions that the equation admits of; that is, if we take the sum of the integral parts of the arithmetical series

$$\frac{(d-c)x'}{b} + \frac{(d-2c)x'}{b} + \frac{(d-3c)x'}{b} + \frac{(d-4c)x'}{b} + \&c.;$$

as also of the arithmetical series

$$\frac{(d-c)y'}{a} + \frac{(d-2c)y'}{a} + \frac{(d-3c)y'}{a} + \frac{(d-4c)y'}{a} + \&c.,$$

the difference of the two will be the whole number of integral solutions; now in each of these series the first and last terms, as also the number of terms, are known, for the general terms being

$$\frac{(d-cz)x'}{b}, \text{ and } \frac{(d-cz)y'}{a},$$

we shall have the extremes by taking the extreme limits of z, that is, $z = 1$, and $z < \dfrac{d-a-b}{c}$, which last value of z also expresses the number of terms in the series.

If, therefore, we find the sums of the two whole series, and then the sum of the fractional parts in each, by deducting these last sums, each from the corresponding whole sum, the sum of the integral parts of each series will be obtained.

In summing the fractional parts, there will be no necessity to go through the whole series; for, as the denominator in each is constant, these fractions will necessarily recur in periods, and the number in each period can never exceed the denominator;* it will therefore only be necessary to find the sum of the fractions in one period, and to multiply this sum by the number of periods in order to get the sum of all the fractions, observing, however, that when there is not an exact number of periods, the overplus fractions must be summed by themselves, which may be readily done, since they will be the same as the leading terms of the first period; it must also be remembered that $\frac{b}{b}$ is to be considered as a fraction in the first series, as in Prob. II.

EXAMPLES.

1. Given the equation $5x+7y+11z=224$, to find the number of solutions which it admits of in positive integers.

$$\text{Here the greatest limit of } z < \frac{224-5-7}{11} \text{ is } 19;$$

also in the equation $5x'-7y'=1$, we have $x'=3$, and $y'=2$,

also $a=5$, and $b=7$;

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* This will appear evident from the following considerations. If in the first series d and c be prime to each other, and neither of them prime to b, each term will be wholly integral; that is, the fractions will be all 0. If b be prime to d, and not to c, the fractions will be all equal. If b be prime to c, and not to d, then the fractions will recur after the first integral term, which can never lie beyond the bth term; and, finally, if a, b, c be all prime to each other, the series of fractions will always recur after the bth term. Similar observations evidently apply to the second series.

therefore, the two series, of which the sums are required, beginning with the least terms, $\frac{(d-19c)x'}{b}$, and $\frac{(d-19c)y'}{a}$ will be

$$\frac{3.15}{7}+\frac{3.26}{7}+\frac{3.37}{7}+\ldots\ldots\frac{3.113}{7},$$

and

$$\frac{2.15}{5}+\frac{2.26}{5}+\frac{2.37}{5}+\ldots\ldots\frac{2.113}{5};$$

the common difference in the first being $\frac{3.11}{7}$, and in the second $\frac{2.11}{5}$, and the number of terms in each 19.

Now the sum of the first series is $928\frac{2}{7}$,

and the sum of the second .... $866\frac{2}{5}$,

also the first period of fractions, in the first series, is

$$\tfrac{3}{7}+\tfrac{1}{7}+\tfrac{6}{7}+\tfrac{4}{7}+\tfrac{2}{7}+\tfrac{7}{7}+\tfrac{5}{7}=4,$$

and the first period in the second series is

$$0+\tfrac{2}{5}+\tfrac{4}{5}+\tfrac{1}{5}+\tfrac{3}{5}=2,$$

$\frac{7}{7}$ being considered as a fraction in the first period, but not $\frac{5}{5}$ in the second.

Hence the number of terms in each series being 19, we have two periods and five terms of the first series $=2\times4+$ the first five fractions $=10\frac{2}{7}$, for the sum of all the fractions; and therefore $928\frac{2}{7}-10\frac{2}{7}=918=$ sum of the integral terms of the first series: also in the second we have three periods and four terms $=3\times2+1\frac{2}{5}=7\frac{2}{5}$; and therefore $866\frac{2}{5}-7\frac{2}{5}=859=$ sum of the integral terms of the second series; whence $918-859=59$ is the whole number of positive integral solutions.

In a similar manner may the number of solutions be obtained when there are four or more unknown quantities.

2. It is required to determine the number of integral solutions that the equation $3x+5y+7z=100$ will admit of.

Ans. 41.

3. It is required to determine the number of integral solutions that the equation $7x+9y+23z=9999$ will admit of.

Ans. 34365.

PROBLEM V.

To find the integral values of three unknown quantities in two equations.

When there are two equations and three unknown quantities, one of the unknowns may be exterminated as in simple equations (Art. 57, Chap. II), and the other unknowns may be found as in Prob. I of the present chapter.

EXAMPLES.

1. Given $\left\{ \begin{array}{r} 2x + 5y + 3z = 51 \\ 10x + 3y + 2z = 120 \end{array} \right\}$, to find all the integral values of x, y, and z.

Here, multiplying the first equation by 5 and subtracting the second, there results $22y + 13z = 135$;

$$\text{whence } z = \frac{135 - 22y}{13} = 10 - y - \frac{9y - 5}{13},$$

$$\begin{array}{l} 13y \\ \underline{9y - 5} \\ 4y + 5 \\ \underline{2} \\ \underline{8y + 10} \\ \underline{y - 15} \end{array}$$

$\therefore y = 2$, and $z = 7$, which are the only values of y and z,

$$\therefore x = 10.$$

It should be remarked here that we are not to expect that, when z and y admit of several values, each will satisfy the proposed

equations: for the corresponding values of x may be fractional. All that we can infer is that the integral values of y and z, deduced as above, contain among them all those which can subsist with integral values of x; but what values do really so subsist can be ascertained only by trying each pair in succession.

2. Given $\left\{\begin{array}{l} 3x + 5y + 7z = 560 \\ 9x + 25y + 49z = 2920 \end{array}\right\}$, to find all the integral values of x, y, and z.

$$\text{Ans.}\left\{\begin{array}{l|l} z = 15 & 30 \\ y = 82 & 40 \\ x = 15 & 50 \end{array}\right.$$

PROBLEM VI.

To find the least whole number which, being divided by given numbers, shall leave given remainders.

Let a, a', a'', &c. be the given divisors, and b, b', b'', &c. the respective remainders; also, call the required number N, then $N = ax + b = a'y + b' = a''z + b'' =$ &c.; therefore $ax - a'y = b' - b$: find now the least values of x and y in this equation, then will $ax + b$, or $a'y + b'$, be the least whole number that fulfils the first two conditions; call this number c, then it is obvious that this, and every other number fulfilling the same conditions, will be contained in the expression $aa'z' + c$,* z' being 0, 1, 2, &c. successively; we have, therefore, $aa'z' + c = a''z + b''$, to find the least values of z' and z, in which case $aa'z + c = a''z + b''$, will be the least whole number fulfilling the first three conditions: call this number d, then will this, and every other number fulfilling the same conditions, be contained in the expression $aa'a''y' + d$, and equating this with the fourth expression for the value of N, and deducing thence the least value of y', the expression $aa'a''y' + d$ will then be the least number answering the first four conditions; and so on to any proposed extent.

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* We are here reasoning on the supposition that a, a' &c. are prime to each other; if, however, they have a common factor, it should be expunged from the expression $aa'z'$.

EXAMPLES.

1. Find the least whole number which, being divided by 11, 19, and 29, shall leave the remainders 3, 5, and 10, respectively.

Here $N = 11x + 3 = 19y + 5 = 29z + 10$,

and $\therefore\ 19y - 11x = -2$, and $y = \dfrac{11x - 2}{19}$;

$$\begin{array}{r}
11x - 2 \\
2 \\
\hline
22x - 4 \\
19x \\
\hline
3x - 4 \\
7 \\
\hline
21x - 28 \\
\hline
x + 24 \\
\hline
\end{array}$$

$\therefore\ x = 14$, and $11x + 3 = 157$; hence we have

$$11 \times 19x' + 157 = 209x' + 157 = 29z + 10,$$

$$\text{and } \therefore\ z = \frac{209x' + 147}{29} = 7x' + 5 + \frac{6x' + 2}{29};$$

$$\begin{array}{r}
6x' + 2 \\
5 \\
\hline
30x' + 10 \\
29x' \\
\hline
x' + 10 \\
\hline
\end{array}$$

$\therefore x' = 19$, and, consequently, $209x' + 157 = 4128$, the number required.

2. Find the least whole number which, being divided by 17 and 26, shall leave for remainders 7 and 13, respectively.

Ans. 143.

3. Find the least whole number which, being divided by 28, 19, and 15, shall leave for remainders 19, 15, and 11, respectively.

Ans. 7691.

4. Find the least whole number which, being divided by 3, 5, 7, and 2, shall leave for remainders 2, 4, 6, and 0, respectively.

Ans. 104.

5. Find the least whole number which, being divided by each of the nine digits, shall leave no remainders.

Ans. 2520.*

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* For several particulars in this chapter the author is indebted to Barlow's 'Theory of Numbers,' a work which cannot be too strongly recommended to the notice of the English student. There is no part of mathematical science that requires such an intimate acquaintance with the properties of numbers as the indeterminate analysis, and the work just mentioned is the only production on that interesting subject, of any extent, in the English language, with the exception of Malcolm's 'Arithmetic.'

CHAPTER X.

ON THE DIOPHANTINE ANALYSIS.

(183.) DIOPHANTINE ALGEBRA* is that part of analysis which relates to the finding of particular rational values for general expressions under a surd form; the principal methods of effecting which are comprehended in the following problems.

PROBLEM I.

To find such values of x as will render rational the expression

$$\sqrt{ax^2+bx+c}.$$

Before we can give any direct investigation of this problem, it will be necessary to consider the nature of the known quantities a, b, c, because there are several cases in which the thing here proposed to be done becomes impossible, and that solely on account of these known quantities.

CASE 1. *When $a=0$, or when the expression is of the form* $\sqrt{bx+c}$.

Put $\sqrt{bx+c}=p$, or $bx+c=p^2$, then $x=\frac{p^2-c}{b}$; consequently, whatever value be given to p, there must necessarily result a corresponding value of x that will render the proposed expression rational, and equal to p.

* So called from Diophantus, a Greek mathematician, who lived about 300 years before Christ, and who appears to have been the first writer on this branch of Algebra.

EXAMPLES.

1. Find a number such that, if it be multiplied by 5 and the product increased by 2, the result shall be a square.

Put $5x+2=p^2$, then $x=\frac{p^2-2}{5}$: if we assume $p=2$, then $x=\frac{2}{5}$; and by assuming other values for p, different values of x may be obtained.

2. Find two numbers, whose difference shall be equal to a given number a, and the difference of whose squares shall be also a square.

Let x be one number, then $a+x$ is the other, and we have to make $(a+x)^2-x^2$, or a^2+2ax, a square.

Put $a^2+2ax=p^2$, then $x=\frac{p^2-a^2}{2a}$, where the value of p may be any number assumed at pleasure.

3. Find a number such that, if it be multiplied by 9 and the product diminished by 7, the result shall be a square.

4. Find a number such that, if it be increased by $\frac{1}{3}$ of its own value and 11 be taken from the sum, the remainder shall be a square.

CASE 2. *When* $c=0$, *or when the expression is of the form* $\sqrt{ax^2+bx}$.

Put $\sqrt{ax^2+bx}=px$, or $ax^2+bx=p^2x^2$, then $ax+b=p^2x$; whence $x=\frac{b}{p^2-a}$, and whatever value be given to p in this expression, there will result a value of x that will make the proposed expression rational.

EXAMPLES.

1. Find a number such that, if its half be added to double its square, the result shall be a square.

Let x be the number, then we must have $2x^2+\frac{1}{2}x=$ a square, which denote by p^2x^2; then $2x+\frac{1}{2}=p^2x$, or $2x-p^2x=-\frac{1}{2}$; $\therefore x=\frac{\frac{1}{2}}{p^2-2}$, p being any number whatever. If p be taken $=2$, then $x=\frac{1}{4}$.

2. Find two numbers, whose sum shall be equal to a given number a, and whose product shall be a square.

Let x be one number, then $a - x$ is the other, and we have to make $ax - x^2$ a square. Put $ax - x^2 = p^2x^2$, then $a - x = p^2x$, whence $x = \frac{a}{p^2+1}$, p being any number whatever.

3. Find a number such that, if its square be multiplied by 7, and the number itself by 8, the sum of the products shall be a square.

4. Find a number such that, if its square be divided by 10 and the number itself by 3, the difference of the quotients shall be a square.

CASE 3. *When c is a square, or when the expression is of the form*

$$\sqrt{ax^2 + bx + c^2}.$$

Put $\sqrt{ax^2 + bx + c^2} = px + c$, then $ax^2 + bx + c^2 = p^2x^2 + 2cpx + c^2$, or $ax^2 + bx = p^2x^2 + 2cpx$, $\therefore$ $ax + b = p^2x + 2cp$, whence $x = \frac{2cp - b}{a - p^2}$.

EXAMPLES.

1. Find two numbers, whose sum shall be 16, and such, that the sum of their squares shall be a square.

Let x be one number, then $16 - x$ is the other, and we have to make $x^2 + (16 - x)^2$, or $2x^2 - 32x + 256$, a square, which denote by $(px - 16)^2 = p^2x^2 - 32px + 256$, and we then have $2x^2 - 32x = p^2x^2 - 32px$, or

$$2x - 32 = p^2x - 32p, \text{ whence } x = \frac{32(p-1)}{p^2 - 2}.$$

If we take $p = 3$, we shall have $x = 9\frac{1}{7}$, $\therefore$ the two numbers are $9\frac{1}{7}$, and $6\frac{6}{7}$.

2. Find two numbers, whose difference shall be equal to a given number a, and the sum of whose squares shall be a square.

CASE 4. *When a is a square, or when the expression is of the form*

$$\sqrt{a^2x^2 + bx + c}.$$

Put $\sqrt{a^2x^2 + bx + c} = ax + p$, or

$$a^2x^2 + bx + c = a^2x^2 + 2pax + p^2,$$

then $bx + c = 2pax + p^2$, $\therefore x = \dfrac{c - p^2}{2pa - b}$.

EXAMPLES.

1. Find a number such that, if it be increased by 2 and 5 separately, the product of the sums shall be a square.

Let x be the number, then we have to make

$(x + 2)\ (x + 5)$, or $x^2 + 7x + 10$, a square, which denote by $(x - p)^2$, then $x^2 + 7x + 10 = x^2 - 2px + p^2$, or $7x + 10 = -2px + p^2$,

$$\therefore x = \frac{p^2 - 10}{7 + 2p}.$$

If we take $p = 4$, we shall have $x = \frac{2}{5}$.

2. Find two numbers, whose difference shall be 14, and such that, if the first be increased by 3 and the second by 4, the product of the sums shall be a square.

3. Find two numbers, whose difference shall be 3, such that, if twice the first increased by 3 be multiplied by twice the second diminished by 3, the product shall be a square.

Ans. $\begin{cases} \text{Any two numbers whatever, whose difference} \\ \text{is 3, the less being considered the } \textit{first}. \end{cases}$

CASE 5. *When neither a nor c is a square, but when $b^2 - 4ac$ is a square.*

In this case it will first be necessary to show that the expression $ax^2 + bx + c$ will always be resolvable into two possible factors. For

if we put $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, and solve the equation, or find the two values of x in it, as $x = k$ and $x = k'$, then $x - k$ and $x - k'$ will obviously be the two factors of $x^2 + \frac{b}{a}x + \frac{c}{a}$; and therefore

$a(x-k)(x-k')$ will be equal to the proposed expression.

Now the values of x in the above equation are

$$x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}, \text{ and } x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a},$$

or putting $b^2 - 4ac = d^2$, the values of x are

$$\frac{d-b}{2a}, \text{ and } -\frac{b+d}{2a}, \text{ and, consequently,}$$

$$(ax + \frac{b-d}{a})(x + \frac{b+d}{2a}) = ax^2 + bx + c;$$

we see, therefore, that the proposed expression under these conditions is always resolvable into two factors.

Let there be then

$$\sqrt{ax^2 + bx + c} = \sqrt{(fx+g)(hx+k)},$$

which put equal to $p(fx+g)$, then

$$(fx+g)(hx+k) = p^2(fx+g)^2;$$

$$\text{or } (hx+k) = p^2(fx+g);$$

$$\text{whence } x = \frac{p^2 g - k}{h - p^2 f}.$$

EXAMPLES.

1. Find such a value of x as will render the expression $6x^2 + 13x + 6$ a square.

Here $a = 6$, $b = 13$, and $c = 6$; and, as this expression evidently does not belong to any of the preceding cases, it will be proper to try whether $b^2 - 4ac$ is a square, which it is found to be, viz. 25: we are certain, therefore, that the expression may be represented by two factors, which are readily found to be $2x + 3$, and $3x + 2$.

Put, therefore, $6x^2+13x+6$, or $(2x+3)(3x+2)=[p(2x+3)]^2$, and it follows that $3x+2=p^2(2x+3)$,

$$\text{whence } x=\frac{3p^2-2}{3-2p^2}.$$

If we take $p=1$, then $x=1$, and the expression becomes equal to 25.

2. Find such a value of x as will make $2x^2+10x+12$ a square.

3. Find such a value of x as will render rational the expression $\sqrt{8x^2+6x-2}$.

CASE 6. *When the proposed expression can be divided into two parts, one of which is a square, and the other the product of two factors.*

This is the last case in which any general method of proceeding can be pointed out, and may be often serviceable when the expression does not come under either of the preceding cases. It is, however, sometimes troublesome to find whether the proposed expression can be decomposed, as this case requires, or not; but if it be ascertained that it can, the expression $\sqrt{ax^2+bx+c}$ may be put under the form $\sqrt{(dx+e)^2+(fx+g)(hx+k)}$, and if we equate this with $(dx+e)+p(fx+g)$, there will result

$$(dx+e)^2+(fx+g)(hx+k)$$
$$=(dx+e)^2+2p(dx+e)(fx+g)+p^2(fx+g)^2,$$
$$\text{or } hx+k=2p(dx+e)+p^2(fx+g);$$
$$\text{whence } x=\frac{p(2e+pg)-k}{h-p(2d+pf)}.$$

EXAMPLES.

1. Find a value of x such, that $2x^2+8x+7$ shall be a square.

This expression, after a few trials, is found to be equivalent to $(x+2)^2+(x+1)(x+3)$, which, being equated with

$[(x+2)-p(x+1)]^2=(x+2)^2-2p(x+2)(x+1)+p^2(x+1)^2$,

there results $x+3=-2p(x+2)+p^2(x+1)$;

$$\text{whence } x=\frac{p^2-4p-3}{1+2p-p^2}.$$

If we take $p = 3$, we shall have $x = 3$, and

$$2x^2 + 8x + 7 = 49.$$

2. Find a value of x such, that $12x^2 + 17x + 6$ may be a square.

(184.) We have now given all the cases in which general methods have been discovered to render the expression $\sqrt{ax^2 + bx + c}$ rational; but as it may have rational values in other cases, it is of importance to be able to determine them.

Now this can be done only when one satisfactory value is already known, which value must therefore be found by trial; this being obtained, other values may be readily deduced.

(185.) Suppose the expression $\sqrt{ax^2 + bx + c}$ is found to become rational when $x = r$, and that the value of the expression in this case is s; then $ar^2 + br + c = s^2$. Put $x = y + r$, and we have, by substitution,

$$ax^2 + bx + c = a(y + r)^2 + b(y + r) + c$$
$$= ay^2 + (2ar + b)y + ar^2 + br + c;$$
$$= ay^2 + (2ar + b)y + s^2,$$

and, as this form comes under Case 3, the value of y, in order that this last expression may be a square, can be found, and thence that of $x = y + r$.

EXAMPLES.

1. Find such values of x that will render the expression $\sqrt{10 + 8x - 2x^2}$ rational.

This expression is found to become rational when $x = 3$.

Put, therefore, $x = 3 + y$, and we have, by substitution, $10 + 8x - 2x^2 = 16 - 4y - 2y^2$, which must be a square; denote it by $(4 - py)^2 = 16 - 8py + p^2y^2$, and we shall have

$$16 - 4y - 2y^2 = 16 - 8py + p^2y^2,$$
$$\text{or} - 4 - 2y = - 8p + p^2y;$$
$$\text{whence } y = \frac{8p - 4}{p^2 + 2}.$$

If we take $p = 2$, then $y = 2$, and $\therefore x = 5$, and the value of the proposed expression is 0.

2. Find such values of x as will render the expression $\sqrt{5x^2+12x+8}$ rational.

3. Find a number such that, if three times itself be taken from three times its square, the remainder increased by 3 shall be a square.

PROBLEM II.

To find such values of x as will render rational the expression

$$\sqrt{ax^3+bx^2+cx+d}.$$

There are but two cases in which a direct solution can be given to this problem. These are the following:

CASE 1. *When the last two terms are absent, or when the expression is of the form*

$$\sqrt{ax^3+bx^2}.$$

Put $\sqrt{ax^3+bx^2}=px$, or $ax^3+bx^2=p^2x^2$, then $ax+b=p^2$;

$$\text{whence } x=\frac{p^2-b}{a}.$$

EXAMPLES.

1. Find a number such that, if three times its cube be added to twice its square, the sum shall be a square.

Here we must make $3x^3+2x^2$ a square; let p^2x^2 be the square, then $3x+2=p^2$, $\therefore x=\dfrac{p^2-3}{2}$.

If we take $p=3$, we have $x=3$, the number required.

2. Find a number such that, if five times its square be taken from three times its cube, the remainder shall be a square.

CASE 2. *When the last term is a square, or when the expression is of the form*

$$\sqrt{ax^3 + bx^2 + cx + d^2}.$$

Put $\sqrt{ax^3 + bx^2 + cx + d^2} = \frac{c}{2d}x + d$;*

then $ax^3 + bx^2 + cx + d^2 = \frac{c^2}{4d^2}x^2 + cx + d^2$,

$$\text{or } ax^3 + bx^2 = \frac{c^2}{4d^2}x^2;$$

$$\therefore ax + b = \frac{c^2}{4d^2};$$

$$\text{whence } x = \frac{c^2 - 4bd^2}{4ad^2}.$$

This solution gives only one value of x, but from this, other values, when possible, may be obtained by the method next following.

When the second and third terms are absent, this method evidently fails.

EXAMPLES.

1. Find such a value of x as will make the expression $3x^3 - 5x^2 + 6x + 4$ a square.

Put $3x^3 - 5x^2 + 6x + 4 = (\frac{3}{2}x + 2)^2 = \frac{9}{4}x^2 + 6x + 4$,

$$\text{then } 3x^3 - 5x^2 = \tfrac{9}{4}x^2, \text{ or } 3x - 5 = \tfrac{9}{4};$$

$$\text{whence } x = \tfrac{29}{12},$$

which value being substituted in the proposed expression, makes it equal to $(\frac{45}{8})^2$.

---

* The expression is assumed equal to $\frac{c}{2d}x + d$, in order that the last two terms in its square may be the same as the corresponding terms in the proposed expression.

2. Find such a value of x as will make $x^3 - x^2 + 2x + 1$ a square.

Ans. $x = 2$.

3. Find a value of x that will make the expression $-5x^3+4x^2-6x+1$ a square.

Ans. $x = -1$.

To these two cases may be added, as in the last Problem, a third, by which other values may be had from one being previously known.

(186.) Suppose it is already known that the expression

$$\sqrt{ax^3 + bx^2 + cx + d}$$

becomes rational when $x = r$, and that the value of the expression then becomes $= s$; that is, let

$$ar^3 + br^2 + cr + d = s^2;$$

then, as in Art. (185), put $x = y + r$, and we have

$$\begin{array}{rrrrl}
ay^3 + & 3ary^2 + & 3ar^2y + & ar^3 & = ax^3 \\
 & by^2 + & 2bry + & br^2 & = bx^2 \\
 & & cy + & cr & = cx \\
 & & & d & = d \\
\hline
ay^3 + & b'y^2 + & c'y + & s^2 & = \square,\text{*}
\end{array}$$

b', c', and s^2, representing the sums of the quantities under which they are respectively placed, therefore the value of y may be determined by last case.

EXAMPLES.

1. The expression $\sqrt{x^3 - x^2 + 2x + 1}$ is found to become rational when $x = 2$: it is required to find another value of x that will answer.

Put $x = y + 2$, then $x^3 - x^2 + 2x + 1 = y^3 + 3y^2 + 8y + 9$; assume this last expression equal to

$$(\tfrac{4}{3}y + 3)^2, \text{ or } \tfrac{16}{9}y^2 + 8y + 9;$$

$$\text{then } y^3 + 3y^2 = \tfrac{16}{9}y^2, \text{ or } y + 3 = \tfrac{16}{9};$$

$$\text{whence } y = -\tfrac{11}{9}, \text{ and } \therefore x = 2 + y = \tfrac{7}{9}.$$

---

* This symbol is used to signify the words *a square*.

2. Find a value of x in the expression $\sqrt{x^3+3} = \square$, besides the case $x = 1$.

Ans. $x = -\frac{23}{16}$.

3. Find a value of x in the expression $\sqrt{3x^3+1} = \square$, besides the case $x = 1$.

Ans. $x = -\frac{5}{16}$.

SCHOLIUM.

There are many cases in the preceding Problem in which the unknown quantity admits of only one rational value, and many more in which the expression is impossible. If any expression can be divided into factors, one of which is a square, this square may be rejected, and the remaining factors only used. Thus, if the expression $ax^3 + bx^2$, or $x^2(ax + b)$, is to be made a square, it will only be necessary to make $ax + b$ a square; also in the expression $x^3 - x^2 - x + 1$, which is equal to $(1 - x)^2(1 + x)$, it will be only necessary to make $1 + x$ a square, in order that the whole expression may be a square.

PROBLEM III.

To find such values of x as will render rational the expression

$$\sqrt{ax^4 + bx^3 + cx^2 + dx + e}.$$

In this Problem there are three cases in which a direct solution can be obtained.

CASE 1. *When both the first and last terms are complete squares, or when the expression is of the form*

$$\sqrt{a^2x^4 + bx^3 + cx^2 + dx + e^2}.$$

Put $a^2x^4 + bx^3 + cx^2 + dx + e^2 = (ax^2 + mx + e)^2 = a^2x^4 + 2amx^3 + (m^2 + 2ae)x^2 + 2mex + e^2$;

then, in order that the terms containing x^3 in this equation may destroy each other, we must make

$$b = 2am, \text{ or } m = \frac{b}{2a},$$

and there will result

$$cx^2 + dx = (m^2 + 2ae)x^2 + 2mex;$$

$$\text{whence } x = \frac{d - 2me}{m^2 + 2ae - c},$$

or substituting for m its equal $\frac{b}{2a}$, we have

$$x = \frac{4a(ad - be)}{b^2 + 4a^2(2ae - c)}.$$

or, since e is found in the proposed expression only in its second power it may be taken either positively or negatively; hence we get another value of x, viz.

$$x = \frac{4a(ad + be)}{b^2 - 4a^2(2ae + c)}.$$

Or this case of the problem may be solved differently by making $d = 2me$, when m will be equal to $\frac{d}{2e}$, instead of $\frac{b}{2a}$, and we shall have

$$bx^3 + cx^2 = 2amx^3 + (m^2 + 2ae)x^2;$$

$$\text{whence } x = \frac{m^2 + 2ae - c}{b - 2am};$$

or, substituting for m its equal $\frac{d}{2e}$, we have

$$x = \frac{d^2 + 4e^2(2ae - c)}{4e(be - ad)}$$

$$\text{or } x = \frac{d^2 - 4e^2(2ae + c)}{4e(be + ad)};$$

this last value being obtained from supposing e negative, as before.

Hence, by employing these two methods, four solutions may be obtained: it must be observed, however, that they all fail when b and d are both 0.

EXAMPLES.

1. It is required to find such a value of x, that the expression $x^4-6x^3+4x^2-24x+16$ may be a square.

Put, according to the first of the above methods,

$$x^4-6x^3+4x^2-24x+16=(x^2-3x-4)^2$$
$$=x^4-6x^3+x^2+24x+16,$$

and there results

$$4x^2-24x=x^2+24x,$$
$$\text{or } 4x-24=x+24;$$
$$\text{whence } x=\tfrac{48}{3}=16.$$

If, according to the second method, we put the expression equal to

$$(x^2+3x-4)^2=x^4+6x^3+x^2-24x+16,$$
$$\text{we have } 6x^3+x^2=-6x^3+4x^2;$$
$$\text{whence } x=\tfrac{1}{4}.$$

By taking 4 ($=e$) positive, each of these solutions gives $x=0$.

2. It is required to find such values of x as will make $x^4-2x^3+2x^2+2x+1$ a square.

Ans. $x=4$, or $-\frac{1}{4}$.

3. It is required to find such values of x as will make $4x^4+3x+1$ a square.

Ans. $x=\frac{3}{4}$, or $-\frac{25}{24}$.

CASE 2. *When the first term only is a square, or when the expression is of the form*

$$\sqrt{a^2x^4+bx^3+cx^2+dx+e}.$$

Put $a^2x^4+bx^3+cx^2+dx+e=(ax^2+mx+n)^2=$

$$a^2x^4+2amx^3+(m^2+2an)x^2+2mnx+n^2;$$

then, in order that the first three terms in this equation may destroy each other, we must make

$$\left.\begin{array}{l} b=2am \\ c=m^2+2an \end{array}\right\} \text{whence} \left\{\begin{array}{l} m=\dfrac{b}{2a} \\ n=\dfrac{c-m^2}{2a}=\dfrac{4a^2c-b^2}{8a^3}, \end{array}\right.$$

we have therefore $dx+e=2mnx+n^2$;

$$\text{whence } x=\frac{n^2-e}{d-2mn};$$

or, substituting for m and n their values as deduced above, we have

$$x=\frac{(4a^2c-b^2)^2-64a^6e}{8a^2[8a^4d-b\,(4a^2c-b^2)]}:$$

When b and d are both 0, this formula fails, the same as in the last case.

EXAMPLES.

1. Required a value of x such, that the expression
$4x^4+4x^3+4x^2+2x-6$ may become a square.

Here $m=1$, and $n=\frac{3}{4}$, therefore

put $4x^4+4x^3+4x^2+2x-6=(2x^2+x+\frac{3}{4})^2=$

$$4x^4+4x^3+4x^2+\tfrac{3}{2}x+\tfrac{9}{16}.$$

and we have $2x-6=\frac{3}{2}x+\frac{9}{16}$;

whence $x=\frac{105}{8}=13\frac{1}{8}$.

2. Required such a value of x, that the expression x^4-3x+2 may become a square.

Ans. $x=\frac{2}{3}$.

3. Required such a value of x, that the expression $x^4-2x^3+4x^2-2x+2$ may be a square.

Ans. $x=\frac{1}{4}$.

CASE 3. *When the last term only is a square, or when the expression is of the form*

$$\sqrt{ax^4+bx^3+cx^2+dx+e^2}.$$

Put $ax^4+bx^3+cx^2+dx+e^2=(mx^2+nx+e)^2=$

$$m^2x^4+2mnx^3+(n^2+2me)x^2+2nex+e^2;$$

then, in order that the last three terms on each side of this equation may destroy each other, we must make

$$\left.\begin{array}{l} d=2ne \\ c=n^2+2me \end{array}\right\} \text{ whence } \left\{\begin{array}{l} n=\dfrac{d}{2e} \\ m=\dfrac{c-n^2}{2e}=\dfrac{4ce^2-d^2}{8e^3}, \end{array}\right.$$

and we shall then have

$$ax^4+bx^3=m^2x^4+2mnx^3,$$

$$\text{or } ax+b=m^2x+2mn,$$

$$\text{whence } x=\frac{2mn-b}{a-m^2};$$

or, substituting for m and n, their values as deduced above, we have

$$x=\frac{8e^3[d(4ce^2-d^2)-8be^4]}{64ae^6-(4ce^2-d^2)^2},$$

which formula fails under the same circumstances as those of the preceding cases.

The first case of this Problem is evidently included in each of the last two cases, and therefore either of the two formulæ last deduced is also applicable to the first case.

EXAMPLES.

1. Find such a value of x as will make the expression

$$5x^4-4x^3+3x^2-2x+1 \text{ a square.}$$

Here $m=1$, and $n=-1$, therefore

$$\text{put } 5x^4-4x^3+3x^2-2x+1=(x^2-x+1)^2=$$
$$x^4-2x^3+3x^2-2x+1,$$

$$\text{and we have } 5x^4-4x^3=x^4-2x^3,$$

$$\text{or } 5x-4=x-2;$$

$$\text{whence } x=\tfrac{2}{4}=\tfrac{1}{2}.$$

2. Find a value of x such, that we may have $2x^4-3x+1=\square$.

Ans. $x=\frac{216}{47}$.

3. Find such a value of x that we may have $22x^4 - 40x^3 - 40x^2 + 64x + 16 = \square$.

Ans. $x = \frac{8}{7}$.

When the proposed expression does not come under either of the above cases, then, as in the preceding Problems, one satisfactory value of the unknown quantity must be discovered by trial, after which, other values, when possible, may be obtained; but in this, as well as in the preceding Problems, there are many expressions in which the unknown quantity admits of only one value, and, in a great many instances, the value is impossible.* We now proceed to show how to find other values from having one value already given.

(187.) Suppose it is already known that the expression

$$\sqrt{ax^4 + bx^3 + cx^2 + dx + e}$$

becomes rational when $x = r$, and that we have

$$ar^4 + br^3 + cr^2 + dr + e = s^2.$$

Assume $y + r = x$, and we have

$$\begin{array}{rcl}
ay^4 + 4ary^3 + 6ar^2y^2 + 4ar^3y + ar^4 & = & ax^4 \\
by^3 + 3bry^2 + 3br^2y + br^3 & = & bx^3 \\
cy^2 + 2cry \quad + cr^2 & = & cx^2 \\
dy \quad + dr & = & dx \\
e & = & e \\
\hline
ay^4 + b'y^3 \quad + \quad c'y^2 + d'y + s^2 & = & \square;
\end{array}$$

the terms in the last line representing the sums of the quantities under which they are respectively placed.

Hence the expression is reduced to a form in which the preceding case will apply, and therefore the value of y, and thence that of x, may be determined.

* It would be impracticable to give in this work a view of all the impossible forms of the expression here treated of: the reader is therefore referred to the work mentioned at the conclusion of last chapter.

EXAMPLES.

1. Find such values of x, that the expression

$$3x^4 + 2x^3 - 5x^2 + 7x - 3 \text{ may be a square.}$$

It appears, upon trial, that if 1 be substituted for x, that the expression will become a square, viz. 4.

Put therefore $x = y + 1$, we have

$$3x^4 + 2x^3 - 5x^2 + 7x - 3 = 3y^4 + 14y^3 + 19y^2 + 15y + 4,$$

which must be made a square; therefore, according to the last case, denote the square by

$$(\tfrac{79}{64}y^2 + \tfrac{15}{4}y + 2)^2 = \tfrac{6241}{4096}y^4 + \tfrac{1185}{128}y^3 + 19y^2 + 15y + 4,$$

and we shall then have

$$3y^4 + 14y^3 = \tfrac{6241}{4096}y^4 + \tfrac{1185}{128}y^3,$$

$$\text{or } 3y + 14 = \tfrac{6241}{4096}y + \tfrac{1185}{128};$$

$$\text{whence } y = \tfrac{19424}{6047},$$

$$\text{and consequently, } x = \tfrac{25471}{6047}.$$

2. Find a value of x that will make $\sqrt{x^4 - 2x^2 + 2}$ rational, besides the case $x = 1$.

Ans. $x = \frac{7}{3}$.

3. Find a value of x such, that the expression

$$22x^4 - 128x^3 + 212x^2 - 64x - 26$$

may be a square, the case $x = 1$ being already known.

Ans. $x = \frac{15}{7}$.*

PROBLEM IV.

To find such values of x as will render rational the expression

$$\sqrt[3]{ax^3 + bx^2 + cx + d}.$$

In this Problem there are likewise only three cases in which a direct solution can be obtained. These are as follow:

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* No methods have yet been discovered for rendering expressions of the above kind rational squares, if the unknown quantity exceed the fourth power; not even when a satisfactory case has been obtained by trial.

CASE 1. *When both first and last terms are cubes, or when the expression is of the form*

$$\sqrt[3]{a^3x^3+bx^2+cx+d^3}.$$

Put $a^3x^3+bx^2+cx+d^3=(ax+d)^3=a^3x^3+3a^2dx^2+3ad^2x+d^3$ and we have

$$bx^2+cx=3a^2dx^2+3ad^2x,$$
$$\text{or } bx+c=3a^2dx+3ad^2;$$
$$\text{whence } x=\frac{3ad^2-c}{b-3a^2d}.$$

EXAMPLES.

1. Find a value of x such that the expression

$$x^3+9x^2+4x+8 \text{ may be a cube.}$$

Put $x^3+9x^2+4x+8=(x+2)^3=x^3+6x^2+12x+8$, and we shall then have

$$9x^2+4x=6x^2+12x,$$
$$\text{whence } x=\tfrac{8}{3}=2\tfrac{2}{3}.$$

2. Find a value of x such that the expression $-125x^3+89x^2+28x+8$ may be a cube.

Ans. $x=\frac{88}{61}$.

3. Find a value of x such that the expression $8x^3+42x^2-8x+27$ may be a cube.

Ans. $x=10\tfrac{1}{3}$.

CASE 2. *When the first term only is a cube, or when the expression is of the form*

$$\sqrt[3]{a^3x^3+bx^2+cx+d}.$$

Put $a^3x^3+bx^2+cx+d=(ax+m)^3=a^3x^3+3a^2mx^2+3am^2x+m^3$, and make

$$3a^2m=b, \text{ or } m=\frac{b}{3a^2},$$

and we shall then have

$$cx+d=3am^2x+m^3,$$

$$\text{whence } x = \frac{m^3 - d}{c - 3am^2}$$

or substituting for m its equal $\frac{b}{3a^2}$, we have

$$x = \frac{b^3 - 27da^6}{(3ca^3 - b^2)9a^3}.$$

EXAMPLES.

1. Find a value of x that will make the expression

$$8x^3 - 4x^2 + 2x - 12 \text{ a cube.}$$

Put $8x^3 - 4x^2 + 2x - 12 = (2x - \frac{1}{3})^3 = 8x^3 - 4x^2 + \frac{2}{3}x - \frac{1}{27}$, and we get

$$2x - 12 = \frac{2}{3}x - \frac{1}{27};$$

$$\text{whence } x = \frac{323}{36}.$$

2. Find a value of x such that the expression $x^3 - 3x^2 + x$ may be a cube. Ans. $x = \frac{1}{2}$.

3. Find a value of x such that the expression $x^3 + 3x^2 + 133$ may be a cube. Ans. $x = 44$.

CASE 3. *When the last term only is a cube, or when the expression is of the form*

$$\sqrt[3]{ax^3 + bx^2 + cx + d^3}.$$

Put $ax^3 + bx^2 + cx + d^3 = (mx + d)^3 = m^3x^3 + 3m^2dx^2 + 3md^2x + d^3$, and make

$$c = 3md^2, \text{ or } m = \frac{c}{3d^2},$$

and there results

$$ax^3 + bx^2 = m^3x^3 + 3m^2dx^2,$$

$$\text{or } ax + b = m^3x + 3m^2d;$$

$$\text{whence } x = \frac{3m^2d - b}{a - m^3},$$

or substituting for m its equal $\frac{c}{3d^2}$, we have

$$x = \frac{(c^3 - 3bd^3)\,9d^3}{27ad^6 - c^3}.$$

EXAMPLES.

1. Required such a value of x that will make the expression

$$2x^3 + 3x^2 - 4x + 8 \text{ a cube.}$$

Put $2x^3 + 3x^2 - 4x + 8 = (-\frac{1}{3}x + 2)^3 = -\frac{1}{27}x^3 + \frac{2}{3}x^2 - 4x + 8$, and we have

$$2x^3 + 3x^2 = -\tfrac{1}{27}x^3 + \tfrac{2}{3}x^2,$$

$$\text{or } 2x + 3 = -\tfrac{1}{27}x + \tfrac{2}{3};$$

$$\text{whence } x = -\tfrac{63}{55}.$$

2. Find a value of x such that the expression $3x^3 + 2x + 1$ may be a cube.

Ans. $x = \frac{36}{73}$.

3. Find such a value of x that will make the expression $3x^3 - 6x^2 + 6x + 1$ a cube.

Ans. $x = -\frac{18}{5}$.

These last two cases are evidently applicable to those forms belonging to Case 1; and therefore, when the first and last terms are both cubes, three solutions may be obtained, one from each case; it must, however, be observed, that they all fail when b and c are both 0.

Having now given all the cases in which a direct solution of the problem can be obtained, it remains to show, as in the preceding problems, how, from having a particular solution, others may be derived from it.

(188.) Suppose the expression

$$\sqrt[3]{ax^3 + bx^2 + cx + d}$$

becomes rational when $x = r$, and that then

$$ar^3 + br^2 + cr + d = s^3.$$

Assume $y + r = x$, and we have

$$\begin{aligned} ay^3 + 3ary^2 + 3ar^2y + ar^3 &= ax^3 \\ by^2 + 2bry + br^2 &= bx^2 \\ cy + cr &= cx \\ d &= d \\ \hline ay^3 + b'y^2 + c'y + s^3 &= \text{a cube.} \end{aligned}$$

The expression is therefore reduced to a form which is resolvable by last case.

EXAMPLES.

1. It is required to find such values for x that the expression $2x^3 - 4x^2 + 6x + 4$ may be a cube.

It appears, upon trial, that $x = 1$ is a satisfactory value; put, then, $x = y + 1$, and the expression becomes

$$2y^3 + 2y^2 + 4y + 8,$$

which put equal to

$$(\tfrac{1}{3}y + 2)^3 = \tfrac{1}{27}y^3 + \tfrac{2}{3}y^2 + 4y + 8,$$

and there results

$$2y^3 + 2y^2 = \tfrac{1}{27}y^3 + \tfrac{2}{3}y^2,$$

$$\text{or } 2y + 2 = \tfrac{1}{27}y + \tfrac{2}{3};$$

$$\text{whence } y = -\tfrac{36}{53};$$

$$\text{and, consequently, } x = \tfrac{17}{53}.$$

2. Find a value of x that will make $x^2 + x + 1$ a cube, besides the case $x = -1$.

Ans. $x = -19$.

3. Find such a value of x that the expression $2x^3 - 1$ may be a cube, besides the cases $x = 1$ and $x = 0$.

Ans. Impossible.

ON DOUBLE AND TRIPLE EQUALITIES.

(189.) In the preceding Problems, the object of our investigations has been to find rational values for expressions under a surd form; and our inquiries have been directed to each expression separately. Questions, however, often occur in the diophantine analysis, that require us to find values for the unknown quantity, or quantities, that shall not only render a single expression a square, cube, &c., but that shall also, at the same time, fulfil similar conditions in one or more other expressions, containing the same unknown quantity or quantities. In the case where two expressions are concerned, it is called a *double equality*, and where there are three expressions, a *triple equality*, &c. The following methods of resolving these equalities will be of service to the student in ordinary cases: but in those instances where the methods here given are found to be insufficient, he must be guided by his own penetration and ingenuity, since no general method of proceeding, that shall be suitable to every case that may occur, can be given.

PROBLEM I.

To resolve the double equality

$$ax + b = \square,$$
$$cx + d = \square.$$

Put $ax + b = p^2$, and $cx + d = q^2$, then, equating the two values of x, which these equations furnish, we have

$$\frac{p^2 - b}{a} = \frac{q^2 - d}{c}, \text{ or } cp^2 - cb = aq^2 - ad;$$

therefore

$$c^2p^2 = caq^2 - cad + c^2b,$$

and, consequently, q must be such a value that the expression

$$caq^2 - cad + c^2b$$

may become a square, which value may be ascertained by one or other of the preceding methods, and thence the value of x may be determined.

PROBLEM II.

To resolve the double equality

$$ax^2 + bx = \square,$$
$$cx^2 + dx = \square.$$

Put $x = \frac{1}{y}$, then, if each equality be multiplied by y^2, there will result the double equality

$$a + by = \square,$$
$$c + dy = \square,$$

which belongs to the preceding Problem.

Or put $ax^2 + bx = p^2x^2$, then $ax + b = p^2x$, and, consequently,

$$x = \frac{b}{p^2 - a}, \text{ and } \therefore cx^2 + dx = c\left(\frac{b}{p^2 - a}\right)^2 + d\left(\frac{b}{p^2 - a}\right) = \square;$$

or, multiplying by the square $(p^2 - a)^2$, it becomes

$$cb^2 - abd + bdp^2 = \square;$$

whence p may be determined, and thence x.

PROBLEM III.

To resolve the double equality

$$ax^2 + bx + c = \square,$$
$$dx^2 + ex + f = \square.$$

Here it will be necessary first to resolve the equality

$$ax^2 + bx + c = \square$$

by Problem 1, and to substitute the value of x, so deduced in the second equality

$$dx^2+ex+f=\square,$$

which will, in consequence, rise to the fourth power, and therefore its solution will belong to Prob. III, p. 316.

PROBLEM IV.

To resolve the triple equality

$$ax+by=\square,$$
$$cx+dy=\square,$$
$$ex+fy=\square.$$

Put

$$ax+by=t^2,$$
$$cx+dy=u^2,$$
$$ex+fy=s^2;$$

then, by eliminating y from the first two equations, we have

$$x=\frac{dt^2-bu^2}{ad-bc};$$

and, by eliminating x from the same equations, we have

$$y=\frac{au^2-ct^2}{ad-bc};$$

therefore, by substituting for x and y, in the third equation, their respective values here exhibited, we shall have

$$\frac{af-be}{ad-bc}u^2-\frac{cf-de}{ad-bc}t^2=\square;$$

or putting $u=tz$, and dividing the expression by the square t^2, there arises the equality

$$\frac{af-be}{ad-bc}z^2-\frac{cf-de}{ad-bc}=\square;$$

from which the values of z may be determined.

Having then found the values of z, we shall have, from the above values of x and y, observing to write tz for u, the following results, viz.

$$x = \frac{d - bz^2}{ad - bc} t^2, \text{ and } y = \frac{az^2 - c}{ad - bc} t^2,$$

where t may be any value whatever.

(190.) The above are the most general methods hitherto discovered for the resolution of double and triple equalities; we may therefore proceed to show the practical application of the foregoing parts of the present chapter to the solution of diophantine questions; but, as was before remarked, the student may expect to meet with cases in which the mode of proceeding must be left, in a great measure, for his own penetration and judgment to suggest. Indeed, the subject on which we are now treating has exercised the ingenuity of some of the most eminent mathematicians of Europe; but Euler and Lagrange have been the most successful in combating the difficulties with which it is attended. The performances of the former are contained in the second volume of his Algebra, which, with the additions of Lagrange, forms the most complete body of information on the diophantine analysis extant; and it is to this work chiefly that the attention of the student is directed.* In the following solutions it will frequently be observed that much depends upon the nature and relation of the assumptions made at the commencement, as a little artifice and ingenuity here will often enable us readily to satisfy one or two conditions of the question, when those that remain may be fulfilled by one or other of the known methods already given.

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* The reader is also referred to Barlow's Theory of Numbers; to Leybourn's Mathematical Repository; to the papers of Mr. Cunliffe, in different volumes of the Gentleman's Mathematical Companion; to a paper by the late Professor Leslie, in vol. ii of the Edinburgh Philosophical Transactions; and to the New York Mathematical Miscellany.

MISCELLANEOUS DIOPHANTINE QUESTIONS.

QUESTION I.

It is required to find a number such, that if it be either increased or diminished by a given number a, and the result be multiplied by the number sought, the product shall, in either case, be a square.

Let x be the number required, then we have to make

$$x^2 + ax = \square,$$
$$x^2 - ax = \square.$$

Put $x = \frac{1}{y}$, then these expressions become

$$\frac{1}{y^2} + \frac{a}{y} = \square,$$
$$\frac{1}{y^2} - \frac{a}{y} = \square;$$

and, multiplying each by y^2, we shall have to make $1+ay$, and $1-ay$, squares; in order to which, put $1 + ay = p^2$, and we get $y = \frac{p^2-1}{a}$, and therefore by substitution,

$$1 - ay = 1 - p^2 + 1 = 2 - p^2 = \square.$$

Now in this last expression we must first find a satisfactory value of p by trial, which is readily effected, since $p = 1$ succeeds: assume, therefore, $p = 1 - q$, and then

$$2 - p^2 = 1 + 2q - q^2 = \square,$$

which denote by

$$(1 - rq)^2 = 1 - 2rq + r^2q^2,$$

and we get

$$2 - q = r^2q - 2r,$$

and, consequently,

$$q = \frac{2r+2}{r^2+1};$$

$$\text{whence } x = \frac{1}{y} = \frac{a}{q^2 - 2q} = \frac{a\,(1+r^2)^2}{4r\,(1-r^2)};$$

where r may be any number whatever; and, should any of the resulting

values of x be negative, they may, with equal truth, be taken positively, as the proposed conditions will evidently obtain in either case.

Suppose $r = 2$, and $a = 1$, then $x = -\frac{25}{24}$, or $+\frac{25}{24}$. If $a = 2$, then $x = \frac{25}{12}$; and so on for other values.

The former part of the above solution might have been conducted differently; thus,

Put $x^2 + ax = p^2x^2$, then $x + a = p^2x$, or

$$x = \frac{a}{p^2 - 1};$$

whence, by substitution,

$$x^2 - ax = \left(\frac{a}{p^2 - 1}\right)^2 - a\left(\frac{a}{p^2 - 1}\right) = \square,$$

or, multiplying by $(p^2 - 1)^2$, we have

$$a^2 - a^2p^2 + a^2 = 2a^2 - a^2p^2 = \square,$$

and dividing by a^2, there results $2 - p^2 = \square$, as before.

QUESTION II.

It is required to find three numbers in arithmetical progression such, that the sum of every two of them may be a square.

Let x, $x + y$, and $x + 2y$ represent the three numbers, and put

$$2x + y = t^2,$$
$$2x + 2y = u^2,$$
$$2x + 3y = s^2:$$

then, exterminating x from the first two of these equations, we obtain

$$\frac{t^2 - y}{2} = \frac{u^2 - 2y}{2},$$

from which we get $y = u^2 - t^2 = s^2 - u^2$, and thence $2u^2 - t^2 = s^2$.

Put now $u = tz$, and this last equation becomes

$$2t^2z^2 - t^2 = s^2,$$

therefore $2z^2 - 1 = \frac{s^2}{t^2}$; hence $2z^2 - 1$ must be a square, which we find to be the case when $z = 1$; therefore, putting $z = 1 - p$, we have

$$2z^2 - 1 = 1 - 4p + 2p^2 = \square,$$

which denote by

$$(1 - rp)^2 = 1 - 2rp + r^2p^2,$$

and we have

$$-4p + 2p^2 = -2rp + r^2p^2,$$

from which we get

$$p = \frac{2r - 4}{r^2 - 2},$$

and thence

$$z = 1 - p = \frac{r^2 - 2r + 2}{r^2 - 2};$$

where r may be any number whatever; and, after having determined z, we shall obtain the values of x and y from the equations

$$x = \tfrac{1}{2}(t^2 - y) = \tfrac{1}{2}(2 - z^2)\,t^2,$$

$$\text{and } y = u^2 - t^2 = (z^2 - 1)\,t^2,$$

t also being any assumed number. In order that x and y may be positive, it is evident that z must lie between 1 and $\sqrt{2}$.

By taking $r = \frac{9}{5}$, we shall have

$$z = \tfrac{41}{31}, \therefore x = \frac{241}{2(31)^2}\,t^2, \text{ and } y = \frac{720}{(31)^2}\,t^2,$$

and making $t = 2 \times 31$, we have $x = 482$, and $y = 2880$; therefore 482, 3362, and 6242, are the numbers required.

QUESTION III.

Find two numbers such that if to each, as also to their sum, a given square, a^2, be added, the three sums shall all be squares.

Let the two numbers be represented by $x^2 - a^2$, and $y^2 - a^2$, and then the first two conditions will be satisfied, and therefore it remains only to make

$$x^2 + y^2 - 2a^2 + a^2, \text{ or } x^2 + y^2 - a^2, \text{ a square,}$$

which denote by m^2; then

$$x^2 - a^2 = m^2 - y^2, \text{ or } (x + a)(x - a) = (m + y)(m - y).$$

Put $x + a = p(m - y)$, then $x - a = \frac{m + y}{p}$,

$$\text{whence } x = p(m - y) - a = \frac{m + y}{p} + a,$$

and, consequently,

$$y = \frac{p^2 m - 2ap - m}{p^2 + 1}.$$

Suppose $a = 1$, $p = 2$, and $m = 8$, then $y = 4$, and $x = 7$.

QUESTION IV.

Find three squares, whose sum shall be a square.

Let the three squares be x^2, y^2, and z^2; then

$$x^2 + y^2 + z^2 = \square.$$

Put $y^2 = 2xz$, or $x = \frac{y^2}{2z}$, and the expression becomes

$$x^2 + 2xz + z^2,$$

which is obviously a square, y and z being any assumed numbers. If we take $y=4$, and $z=8$, then $x=1$, and

$$1+16+64=81.$$

Otherwise, assume

$$x^2+y^2+z^2=(x+p)^2=x^2+2px+p^2,$$

and we shall then have

$$x=\frac{y^2+z^2-p^2}{2p}.$$

If we take $y=4$, $z=8$, and $p=8$, we shall have $x=1$, as before.

If we take $y=4$, $z=12$, and $p=10$, then $x=3$, &c.

QUESTION V.

Find three square numbers, whose sum shall be equal to a given square number a^2.

Here we have

$$x^2+y^2+z^2=a^2.$$

Put $y^2=2xz$, and we have

$$x^2+2xz+z^2=a^2;$$

therefore $x+z=a$, or $x=a-z$; and, by substitution,

$$y^2=2az-2z^2$$

which denote by p^2z^2, and we obtain

$$2a-2z=p^2z, \text{ whence } z=\frac{2a}{p^2+2};$$

$$\text{therefore } x=a-\frac{2a}{p^2+2}.$$

If we take $a=9$, and $p=4$, then $z=1$, $x=8$, and $y=4$.

Hence the three squares are 1, 16, and 64: and

$$1+16+64=9^2.$$

QUESTION VI.

Find four numbers such that if their sum be multiplied by any one, increased by unity, the products shall all be squares.

Let w^2-1, x^2-1, y^2-1, and z^2-1, be the four numbers; then all the conditions will be fulfilled, if we make

$$w^2+x^2+y^2+z^2-4=\square;$$

in order to which, put $w^2=4$, then there only remains to make $x^2+y^2+z^2=\square$, which has been already done, Quest. IV; and x, y, and z, may be 3, 4, and 12, respectively: hence the required numbers are 3, 8, 15, and 143.

QUESTION VII.

It is required to divide a number that is equal to the sum of two known squares, a^2 and b^2, into two other square numbers.

Let x^2 and y^2 represent the required squares; then

$$a^2+b^2=x^2+y^2,$$

$$\text{or } a^2-y^2=x^2-b^2;$$

$$\text{that is } (a+y)(a-y)=(x+b)(x-b).$$

Put $a+y=p(x-b)$, then $a-y=\dfrac{x+b}{p}$, whence

$$y=p(x-b)-a=a-\frac{x+b}{p}, \text{ from which we get}$$

$$x=\frac{bp^2+2ap-b}{p^2+1}.$$

Suppose $a=2$, and $b=9$, and assume $p=2$, then we have

$$x=7, \text{ and } y=p(x-b)-a=-6,$$

so that in this case the two required squares are 49 and 36.

QUESTION VIII.

Find three square numbers in arithmetical progression.

Let x^2, y^2, and z^2 represent the three required squares,

$$\text{then } x^2 + z^2 = 2y^2, \text{ and}$$

$$2x^2 + 2z^2 = 4y^2 = \square.$$

Put $x = m + n$, and $z = m - n$, and we have

$$4m^2 + 4n^2 = 4y^2, \text{ or } m^2 + n^2 = y^2:$$

now this last condition is fulfilled by making

$$m = p^2 - q^2, \text{ and } n = 2pq;*$$

therefore, substituting these values of m and n in the above expressions for x and z, we have

$$x = p^2 - q^2 + 2pq,$$

$$z = p^2 - q^2 - 2pq,$$

$$y = p^2 + q^2,$$

p and q being any numbers whatever.

If we take $p = 2$, and $q = 1$, we shall have

$$x = 7, y = 5, \text{ and } z = 1;$$

that is, the three squares will be 7^2, 5^2, and 1^2.

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* It is obvious that $(p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2$, also

$$(p^2 + q^2)^2 - (2pq)^2 = (p^2 - q^2)^2, \text{ or}$$

$$(p^2 + q^2)^2 - (p^2 - q^2)^2 = (2pq)^2;$$

hence two square numbers may always be readily found, such that their sum or their difference shall be a square.

QUESTION IX.

Find four numbers such that their sum shall be a square; also, if their sum be multiplied by any one of them, and the product be increased by unity, the results shall be all squares.

Let $x-1$, $x+1$, $x-y$, and $x+y$, represent the four numbers; then we have to make

$$4x = \square,$$
$$4x^2 - 4x + 1 = \square,$$
$$4x^2 + 4x + 1 = \square,$$
$$4x^2 + 4xy + 1 = \square,$$
$$4x^2 + 4xy + 1 = \square:$$

now the second and third of these expressions are already squares. It only remains, therefore, to make the other three squares. Assume $x = 4$, then the first expression becomes a square, and the fourth and fifth become $65 - 16y$, and $65 + 16y$; put the first of these $= m^2$, and we get

$$y = \frac{65 - m^2}{16}; \text{ put the second } = n^2, \text{ and we get}$$

$$y = \frac{n^2 - 65}{16};$$

whence $65 - m^2 = n^2 - 65$, or

$n^2 = 130 - m^2$, which evidently obtains when $m = 3$, when we have $n = 11$; therefore $y = 3\frac{1}{2}$, and, consequently, the three numbers are 3, 5, $\frac{1}{2}$, and $7\frac{1}{2}$.

QUESTION X.

Find three cube numbers, whose sum shall be a cube.

Let x^3, y^3, and z^3, represent the three cubes, and put their sum $= (x+z)^3 = x^3 + 3x^2z + 3xz^2 + z^3$, and their results

$$y^3 = 3x^2z + 3xz^2.$$

Put now $x = pz$, and then

$$y^3 = 3p^2z^3 + 3pz^3,$$

$$\text{whence } 3p^2 + 3p = \text{a cube};$$

therefore we have to find, by trial, a satisfactory value of p, which presents itself in the case $p = \frac{1}{8}$; consequently, if we make $z = 8$, we get $x = pz = 1$; whence $y = 6$, and the three cubes are 1^3, 6^3, and 8^3, whose sum is 9^3: and by making $z =$ any multiple of 8, we may obtain as many integral solutions as we please.

QUESTION XI.

Find three numbers in arithmetical progression such that the sum of their cubes may be a cube.

Let $a - x$, a, and $a + x$, represent the three required numbers; then the sum of their cubes is $3a^3 + 6ax^2$, which must be a cube; or putting $x = \frac{a}{p}$, we have $3a^3 + 6\frac{a^3}{p^2} =$ a cube, therefore $3 + \frac{6}{p^2} =$ a cube; and if we now put $p^2 = 2n^3$, this last expression will become

$$\frac{3n^3 + 3}{n^3}; \text{ whence } 3n^3 + 3 = \text{a cube},$$

therefore it remains to satisfy the following conditions, viz.

$$2n^3 = p^2,$$

$$3n^3 + 3 = \text{a cube};$$

the first is readily effected by assuming $p = 2nq$, or $2n^3 = 4n^2q^2$, which gives $n = 2q^2$; and, by substitution, the second becomes

$$24q^6 + 3 = \text{a cube},$$

in which a satisfactory value of q immediately presents itself, viz. $q = 1$, which value gives $n = 2$, and $p = 4$; therefore, assuming $a = 4$, we have $x = 1$, and the three required numbers are 3, 4, and 5, which give $3^3 + 4^3 + 5^3 = 6^3$.

If we take $a = 8$, then $x = 2$, and the numbers are 6, 8, and 10, which give $6^3 + 8^3 + 10^3 = 12^3$, and taking a any other multiple of 4, we may obtain as many integral solutions as we please.

QUESTION XII.

Find three square numbers in arithmetical progression such that if the root of each be increased by 2, the three sums may be all squares, of which the sum of the first and third shall be also a square.

By Question VIII, the general expressions for the roots of three squares in arithmetical progression are

$$p^2+2pq-q^2,$$
$$p^2+q^2$$
$$p^2-2pq-q^2,$$

or by taking $q=1$, these expressions become

$$p^2+2p-1,$$
$$p^2+1,$$
$$p^2-2p-1;$$

and adding 2 to each of these, according to the question, we have

$$p^2+2p+1=\square,$$
$$p^2+3 \quad =\square,$$
$$p^2-2p+1=\square;$$

also, adding the first and third of these expressions together,

$$2p^2+2=\square;$$

therefore, since the first and third expressions are already squares, it only remains to make

$$p^2+3=\square,$$
$$2p^2+2=\square,$$

which they will evidently be when $p=1$: put then $p=m+1$, and we have to make

$$m^2+2m+4=\square,$$
$$2m^2+4m+4=\square.$$

In order to effect this, assume the second expression

$$=(nm+2)^2=n^2m^2+4nm+4,$$

and there results

$$2m^2 + 4m = n^2m^2 + 4nm,$$

from which we obtain $m = \frac{4(1-n)}{n^2-2}$; and by substituting this value of m in the first expression, we shall have

$$16\left(\frac{1-n}{n^2-2}\right)^2 + 8\left(\frac{1-n}{n^2-2}\right) + 4 = \square,$$

or multiplying by $\frac{(n^2-2)^2}{4}$, and adding together the like terms, we have

$$n^4 - 2n^3 + 2n^2 - 4n + 4 = \square;$$

assume this expression

$$= (n^2 - n + \tfrac{1}{2})^2 = n^4 - 2n^3 + 2n^2 - n + \tfrac{1}{4},$$

and we shall then have

$$-4n + 4 = -n + \tfrac{1}{4}, \therefore n = \tfrac{5}{4},$$

consequently,

$$m = \frac{4(1-n)}{n^2-2} = \tfrac{16}{7}, \text{ and } p = m + 1 = \tfrac{23}{7};$$

therefore the three required squares are $(\frac{802}{49})^2$, $(\frac{578}{49})^2$, and $(\frac{158}{49})^2$, which are in arithmetical progression, the common difference being $\frac{309120}{49^2}$; and if we increase the root of each by 2, we shall have the three squares $(\frac{30}{7})^2$, $(\frac{26}{7})^2$, and $(\frac{16}{7})^2$, of which the sum of the first and third is the square $(\frac{34}{7})^2$.

13. Find two whole numbers x and y, the least possible, such that their sum and difference shall both be squares.

Ans. 4 and 5.

14. Find two square numbers such that if each be increased by the root of the other, the sums shall both be squares.

Ans. $\frac{1}{36}$ and $\frac{1}{144}$.

15. Find two numbers such that if the square of each be added to their product, the sums shall both be squares.

Ans. 9 and 16.

16. Find two fractions such that if either of them be added to the square of the other, the sums may be equal, and that the sum of their squares may be a square number.

Ans. $\frac{3}{7}$ and $\frac{4}{7}$.

17. Find two numbers such that if their product be added to the sum of their squares, the result may be a square.

Ans. 3 and 5.

18. Find three numbers such that if to the square of each the product of the other two be added, the results shall be all squares.

Ans. 9, 73, and 328.

19. Find two numbers such that their sum, the sum of their squares, and the sum of their cubes, may be all squares.

Ans. 184 and 345.

20. Find three numbers such that their product, increased by unity, shall be a square, also the product of any two increased by unity shall be a square.

Ans. 1, 3, and 8.

21. Find three numbers, whose sum shall be a square, such that if the square of the first be added to the second, the square of the second to the third, and the square of the third to the first, the sums shall be all squares.

Ans. $\frac{39}{80}$, $\frac{1}{40}$, and $\frac{1}{20}$.

22. Find three numbers in arithmetical progression, such that the sum of every two may be a square.

Ans. $120\frac{1}{2}$, $840\frac{1}{2}$, and $1560\frac{1}{2}$.

23. Find three numbers in geometrical progression, such that the difference of every two may be a square.

Ans. 567, 1008, and 1792.

24. Find three square numbers such that the sum of every two may be a square.

Ans. 44^2, 117^2, and 240^2.

25. Find three square numbers such that the difference of every two may be a square.

Ans. 153^2, 185^2, and 697^2.

26. Find three square numbers in geometrical progression, such that if any one of them be increased by its root, the sum shall be a square.

Ans. $(\frac{49}{480})^2$, $(\frac{98}{480})^2$, and $(\frac{196}{480})^2$.

27. Find three square numbers that shall be in harmonical proportion.

Ans. 1225, 49, and 25.

28. Find two numbers such that their sum shall be equal to the sum of their cubes.

Ans. $\frac{5}{7}$, and $\frac{8}{7}$.

29. Find three cubes such that if unity be subtracted from each, the sum of the remainders shall be a square.

Ans. $(\frac{17}{13})^3$, $(\frac{28}{13})^3$, and 2^3.

30. Find two numbers such that their sum shall be a square, their difference a cube, and the sum of their squares a cube.

Ans. 23958, and 34606.

31. Find four numbers such that the product of any three increased by unity shall be a square.

Ans. $\frac{1}{2}$, 2, 3, and $\frac{16016}{625}$.

APPENDIX.

ON PROBABILITIES AND LIFE ANNUITIES.

PERMUTATIONS AND COMBINATIONS.

(1.) PERMUTATIONS are the different arrangements which quantities admit of as respects the order in which they succeed one another, when placed side by side. Thus the quantities a, b admit of two permutations; for they furnish the two arrangements ab and ba; the quantities a, b, c admit of six permutations, as exhibited in the following arrangements, viz.

$$abc, acb, bac, bca, cab, cba.$$

In these instances, each group of quantities consists of the same things differently disposed, all the given quantities entering into each. But it is often required to ascertain the permutations that certain selections from a proposed set of quantities admit of; as, for instance, the permutations of all the *pairs*, of all the groups of *three*, of *four*, &c. Thus the permutations of a, b, c, taken in pairs, or two and two, are

$$ab, ba, ac, ca, bc, cb,$$

being the same in number as when taken all together. These simple illustrations will serve to render the object of the following general problem intelligible.

(2.) PROBLEM I. To find the number of permutations that can be formed out of n quantities, when m of them are taken together.

Let the quantities be represented by a, b, c, d, &c. And first, let them be taken in *pairs*; that is, let $m = 2$. It is plain that the number of those pairs in which a stands first, will be $n - 1$; because each of the remaining $n - 1$ quantities may be written beside a, in succession. For a similar reason, the number of pairs in which b stands first, is also $n - 1$; and so of the others: and as there are altogether n quantities, the total number of permutations, when the quantities are taken two and two, is $n(n - 1)$.

Again: let them be taken in *threes*; that is, let $m = 3$. Suppose one of the n quantities to be suppressed, as a, leaving only $n - 1$ quantities. If these be arranged in pairs, the number of permutations, as we have just seen, will be $(n - 1)(n - 2)$; and if the suppressed a be now prefixed to each of these, we shall thus have $(n - 1)(n - 2)$ permutations of *threes*, each commencing with a. Similarly, by suppressing b, then c, then d, &c., and proceeding as before, we shall, in the whole, have as many times $(n - 1)(n - 2)$ permutations as there are letters a, b, c, &c.; that is n times: therefore the total number of permutations, when the quantities are taken three and three, is $n(n - 1)(n - 2)$.

It is obvious that by imitating this procedure, we shall have $n(n - 1)(n - 2)(n - 3)$ for the number of permutations when the quantities are taken *four* at a time. And, generally, when they are taken m at a time, the number of permutations will be

$$n(n - 1)(n - 2)(n - 3) \ldots\ldots (n - \overline{m - 1}) \ldots\ldots\ldots\ldots [A].$$

It follows from this that if *all* the n quantities are to occur in each group, the number of permutations—writing the factors in reverse order—will be the product of the n consecutive numbers

$$1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots\ldots\ldots\ldots\ldots n.$$

It was observed at (1) that the number of permutations of three quantities, taken two at a time, is equal to the number furnished by taking all three together. It is now easy to see that this property is general; that is, the number of permutations furnished by n quantities, taken all altogether, is equal to the number obtained by taking them $n - 1$ at a time. For if in the above general formula we make m equal to $n - 1$, the last factor $(n - \overline{m - 1})$, will become simply 2; so that when the factors are written in reverse order, we shall have

the product of the n consecutive numbers just exhibited; showing that the permutations are the same in number, whether the n quantities be taken all together, or only $n - 1$ at a time.

(3.) We have now to remark that certain of the n quantities, out of which the permutations are to be formed, may be identical; in which case, certain of the permutations will recur, differing in no respect from one another. In many inquiries it is requisite to retain only those permutations which really differ; hence we have:

PROBLEM III. To find the number of different permutations of n quantities, taken all together, when certain of these n quantities are identical.

As before, we shall begin with the simplest case: Suppose only two of the n quantities are alike: then, as these two enter all the permutations, they enter $1 . 2 . 3 . 4 \ldots\ldots n$ times; and every single permutation furnishes a second by simply interchanging these two identical quantities, by which interchange, however, no real difference is produced: hence the total number of permutations is double the number of those that are really different; to obtain these, therefore, we must divide the total number by 2.

In like manner, if three of the n quantities are identical, the entire series of permutations will obviously be divisible into as many recurring groups of permutations as these three quantities admit of interchanges, or *permutations*. As we require the number of permutations in only *one* of these recurring groups, it is plain that we must divide the number in the entire series by the number of groups; that is by $1 . 2 . 3$, the permutations due to the identical quantities. And generally, if p of the n quantities are identical, there must be as many recurring groups as there are permutations of these p; so that the number of different permutations of n quantities taken all together, when p of them are identical, is

$$\frac{1 . 2 . 3 . 4 \ldots\ldots n}{1 . 2 . 3 \ldots\ldots p}$$

If, besides these, q others are also identical, then in like manner, each of the former groups becomes separable into smaller groups, which recur, as before, as often as there are permutations in these q quantities, that is $1 . 2 . 3 \ldots\ldots q$ times. Hence, when p of the n quantities are

alike, and also q others, the number of differing permutations, formed by taking all together, is

$$\frac{1 \,.\, 2 \,.\, 3 \,.\, 4 \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, n}{(1 \,.\, 2 \,.\, 3 \,\ldots\, p)\,(1 \,.\, 2 \,.\, 3 \,\ldots\, q)}$$

and so on to any extent.*

(4.) It is plain that if we suppress the factors common to numerator and denominator of this fraction, and then arrange the remaining factors of the numerator in reverse order, the expression just deduced may be written in either of the two following forms, viz.

$$\left.\begin{array}{c} \dfrac{n\,(n-1)(n-2)\ldots\ldots(p+1)}{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\ldots\, q} \\ \dfrac{n\,(n-1)\,(n-2)\ldots\ldots(q+1)}{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\ldots\, p} \end{array}\right\} \ldots\ldots\ldots\ldots \text{[B]}$$

which are therefore identical.

If $p + q = n$, that is, if the n quantities consist of two sets of identical quantities, p in one set and q in the other, then $p+1 = n - \overline{q-1}$, and $q + 1 = n - \overline{p-1}$; and the two forms just written will be

$$\left.\begin{array}{c} \dfrac{n\,(n-1)\,(n-2)\,\ldots\ldots\,(n-\overline{q-1})}{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\ldots\, q} \\ \dfrac{n\,(n-1)\,(n-2)\,\ldots\ldots\,(n-\overline{p-1})}{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\ldots\, p} \end{array}\right\} \ldots\ldots \text{[C]}$$

In the first of these the student will recognize the coefficient of the $(q+1)$th term of the development of the binomial $(a+b)^n$, and in the second, the coefficient of the $(p+1)$th term of the same development. It is obvious that these are identical, because the $(p+1)$th coefficient from the beginning is the $(q+1)$th from the end, (page 192.)

As an illustration of these formulas, let it be required to find the number of different permutations that can be formed of the letters in the word *Algebraical*. The total number of letters is eleven, of which the letter a occurs three times, and the letter l twice; therefore the two

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* If $p = q = r = s$, &c., and these separate parcels of like things be k in number, the denominator of the fraction for the different permutations will be $(1 \,.\, 2 \,.\, 3 \,\ldots\ldots\, p)^k$; or, using the numerator in the form [B], the denominator will be $(1 \,.\, 2 \,.\, 3 \,\ldots\ldots\, p)^{k-1}$.

forms [B] give $\frac{11.10.9.8.7.6.5.4}{1.2}$ and $\frac{11.10.9.8.7.6.5.4.3}{1.2.3}$, that is 3326400 for the number of different permutations.

Again; let the number of different permutations that can be formed of the letters in the word *Annan* be required. The total number of letters is five, of which n occurs three times and a twice; therefore the two formulas [C] give $\frac{5.4.3}{1.2.3}$ and $\frac{5.4}{1.2}$, or 10 for the number of different permutations.

It is plain that in each of the forms [C], the number of factors in numerator and denominator is the same, the number being q in the first form and p in the second; we should of course always employ that form in which this number is the smallest, writing down the factors $1.2.3.$ &c. in the denominator up to this smallest number, and then, in succession, over these, as many of the numbers $n.(n-1).(n-2)$ &c. for the numerator.

(5.) Sometimes permutations are to be formed of m things out of n, when liberty is given to repeat any of the n things as often as we please in the permutations, consistently of course with the restriction that only m individuals are to enter each: as, for instance, it might be required to ascertain how many different numbers, consisting of two places of figures, can be expressed by employing no other digits than the four 1, 2, 3, 4. Here repetitions of the same figure not being forbidden, we have the permutations 11, 22, 33, 44, besides those implied in the formula [A], which are 12, thus making in the whole $16 = 4^2$.

If we represent the n things out of which these permutations are to be formed by letters, connected together by the plus or minus sign, as $a + b + c + d + \ldots\ldots\ldots\ldots$, it is obvious that the quantity thus expressed, if multiplied by itself, will furnish all the possible permutations, two and two; and if these be now multiplied by the same multiplier, there will be exhibited all the permutations, three and three; another multiplication will produce all the permutations four and four, and so on. Therefore the number of permutations out of n things taken two and two, allowing repetitions, is . . n^2,

. . . three and three . . n^3,

. . . four and four . . n^4,

. . . m and m . . . n^m.

(6.) COMBINATIONS are the different arrangements which quantities admit of under the restriction that in no two shall the quantities be all the same. In permutations, as we have just seen, it is only necessary that two groups differ in arrangement: in combinations, the mere arrangement of the quantities in the several groups is of no moment; they must each differ by at least one quantity not to be found in any of the other groups. As we have just seen, four quantities, *a*, *b*, *c*, *d*, admit of six *permutations*, but of only one *combination*, viz., *abcd*, or *acbd*, &c., which present the same combination under a different arrangement. If only *three* of the four quantities be taken, the number of combinations will be four, viz.

abc, *abd*, *bcd*, *acd*.

PROBLEM IV. To find the number of combinations that can be formed out of n quantities, when m of them are taken together.

As before, let the quantities be represented by a, b, c, d, &c. And first let them be taken *two* at a time. The number of *permutations* will be expressed by $n(n-1)$, as already shown, and it is plain that each permutation is accompanied by another, consisting of the *same* letters, as, for instance, *ab* and *ba*, *bc* and *cb*, &c., that is for every combination there are two permutations: hence the number of combinations, when the quantities are taken two and two, is $\frac{n(n-1)}{2}$.

Again: let them be taken in *threes*; then the number of permutations will be expressed by $n(n-1)(n-2)$. And for every combination of three, there are $1.2.3$ permutations (p. 346), that is, there are six times as many permutations as combinations, so that the number of combinations, when the quantities are taken three at a time, is $\frac{n(n-1)(n-2)}{1.2.3}$.

In like manner, when they are taken *four* at a time—since, for every combination of four, there are $1.2.3.4$ permutations, and that the total number of permutations is $n(n-1)(n-2)(n-3)$—it follows that the number of combinations, when the n quantities are taken four at a time, is $\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$. And generally, that when the quantities are taken m at a time, the number of combinations are

$$\frac{n(n-1)(n-2)(n-3)\ldots\ldots\ldots(n-\overline{m-1})}{1.2.3.4\ldots\ldots\ldots m},$$

which, as we have before seen, also expresses the number of different *permutations* which can be formed with n quantities taken all together, when these n quantities are composed of two sets: a set of m all alike, and the remaining set of $n-m$ all alike. It is obvious, from the formulas [C], that the expression just deduced will be unaltered in value, though $n-m$ be substituted for m, from which circumstance, the arithmetical calculation may always be shortened, when $m > \frac{1}{2}n$. Thus, suppose it were required, to calculate the number of combinations of 96 things out of 100; then $n=100$ and $m=96$, and the computation by the preceding formula would be very laborious; but from the property just noticed, we may replace m by $100-96$ or 4; so that there will be but four factors in the denominator: and as there are necessarily the same number in the numerator, the expression will be

$$\frac{100 \, . \, 99 \, . \, 98 \, . \, 97}{1 \, . \, 2 \, . \, 3 \, . \, 4} = 3921225$$

for the number of combinations sought.

PROBLEM V. If n quantities be given, as also n' other quantities, to find the number of combinations furnished by the two sets, under the restriction that each group is to contain m quantities out of the first set, and m' out of the second.

The number of combinations of quantities taken, m at a time, out of the first set, is by last problem,

$$\frac{n(n-1)(n-2) \ldots\ldots (n-\overline{m-1})}{1 \, . \, 2 \, . \, 3 \ldots\ldots m},$$

and of quantities taken m' at a time out of the second set,

$$\frac{n'(n'-1)(n'-2) \ldots\ldots (n'-\overline{m'-1})}{1 \, . \, 2 \, . \, 3 \ldots\ldots m'},$$

and as each of these combinations may be combined with every one of the former, the total number will be

$$\frac{n(n-1)(n-2) \ldots\ldots (n-\overline{m-1})}{1 \, . \, 2 \, . \, 3 \ldots\ldots m} \times \frac{n'(n'-1)(n'-2) \ldots\ldots (n'-\overline{m'-1})}{1 \, . \, 2 \, . \, 3 \ldots\ldots m'},$$

which formula it is easy to generalize. If $m = m' = m'' = \&c. = 1$, the number of combinations is $n \, n' \, n''$ &c.

PROBABILITIES.

(7.) It is but seldom that we can predict any of the events of futurity with absolute *certainty*. What we call by this name is in general but a high degree of expectation, which is always susceptible of an accurate evaluation when the operating or controlling circumstances, *as far as we can know them*, are given. From a due consideration of these, the force or degree of our expectation and its defect from absolute certainty may be accurately estimated by a numerical measure, the determination of which is the business of the doctrine of PROBABILITIES: a doctrine which thus has its rise in our ignorance of operating causes, and which would have no existence if this ignorance were removed. The student must never lose sight of this truth, for it is that on which the entire theory is constructed, and in reference to which alone that the terms *chance* and *probability* have any meaning.

In order to the numerical evaluation to which we have adverted, it is necessary that some numerical measure be assumed for *certainty*. Any number may be chosen for this purpose, but as *unity* has several advantages over other numbers, this is accordingly assumed to represent certainty; and thus probabilities of all degrees become expressed by fractions ranging between 0 and 1.

(8.) To give an idea of these representations, let us suppose a common die—a solid of six equal faces—to be thrown upon a table:—which of the six sides will be uppermost?

In our ignorance of the determining causes we have no reason for answering in favour of one face, more than another, that is, any one face is as likely to be uppermost as any other: the probabilities, or chances, are *equal*. We may find the value or amount of each of these equal probabilities from the following considerations. Suppose six persons, each to predict the turning up of a different face; as just stated, each person has the same chance of being right. Suppose a sum of money, S, to become due to him whose prediction is fulfilled. Before the dice is thrown each of these six persons has the same interest in the sum S. If these interests were all to be purchased by another party at their exact value, the price paid for them ought to be exactly S, since one of the six chances must necessarily be successful, and thus the sum S must necessarily be recovered. We thus see in a

moment the exact value of all the six chances together, and therefore, since they are all equal, the value of any one must be a sixth part of this; viz. $\frac{1}{6}$ S. If we were *certain* that any specified face would turn up, the value of the throw would be 1 S; as it is, the value is only $\frac{1}{6}$S. And as *certainty* is 1, the *probability* of an assigned face turning up is $\frac{1}{6}$.

The value of each one of the six chances being $\frac{1}{6}$ S the value of any two must be $\frac{2}{6}$ S, of any three, $\frac{3}{6}$ S, &c.; that is, the probability that one out of any *two* specified faces will turn up is $\frac{2}{6}$, that one out of *three* will turn up $\frac{3}{6}$, one out of *four* $\frac{4}{6}$, one out of *five* $\frac{5}{6}$, one out of *six* $\frac{6}{6}$, or certainty: the probability of the predicted event happening is thus equal to the number of chances *for* the happening, divided by *all* the chances for and against. In the case here considered, all the chances or possible events are six; but it is obvious that the reasoning would have been similar if the number had been ten, twelve, or any other number; so that generally:—

I. *If there are n possible chances, or events, or ways of happening, all equally likely, of which m are favorable to any predicted result, the probability that the prediction will be fulfilled is* $\frac{m}{n}$. *And consequently the probability that it will fail—being what the former probability wants of certainty—must be* $1 - \frac{m}{n}$ *or* $\frac{n-m}{n}$.

(9.) We shall give a few examples of the application of this general principle.

EXAMPLE I. A bag contains 16 balls; 10 black and 6 white: What is the probability that in drawing two of them at once they shall both be white?

By the foregoing principle we must first ascertain all the possible ways in which two balls can be drawn; then, how many of these are favorable to the proposed result:—the latter number divided by the former will express the probability of that result happening.

All the possible combinations of two out of sixteen (6) are $\frac{16 \cdot 15}{1 \cdot 2} = 120 =$ all the ways of happening.

All the possible combinations of two out of six are $\frac{6 \cdot 5}{1 \cdot 2} = 15 =$ all the ways favorable to the result mentioned.

$$\therefore \frac{15}{120} = \frac{1}{8}, \text{ the probability required.}$$

In like manner, $\frac{10 \cdot 9}{16 \cdot 15} = \frac{3}{8}$, the prob. of drawing two black balls.

Consequently, $1 - \frac{1}{8} = \frac{7}{8}$, the prob. of *not* drawing two white balls,

and $1 - \frac{3}{8} = \frac{5}{8}$, the prob. of *not* drawing two black balls.

That is, the probability that one *at least* will be black is $\frac{7}{8}$,

and the probability that one *at least* will be white is $\frac{5}{8}$.

The ratio of the probabilities for and against drawing two white balls is $\frac{1}{8}$ to $\frac{7}{8}$, or 1 to 7: the *odds*, as it is called, *against* the drawing being 7 to 1. In like manner, the *odds* against drawing two black balls is 5 to 3: the odds against any event being always as the probability of failure to the probability of success.

It is obvious that all the preceding probabilities remain the same whether the two balls be drawn at once, as in the question, or in succession: it can make no difference when the hand is in the bag whether the two balls are taken up simultaneously or one after the other; nor in the latter case, whether the hand is retained in the bag till the second ball is taken up, or withdrawn and then inserted again.

2. Two halfpence are tossed in the air; what is the probability that they shall turn up heads?

In the two halfpence there are four sides or faces; and all the combinations of two out of four are $\frac{4 \,.\, 3}{1 \,.\, 2} = 6$: but in neither halfpenny can its own sides be combined with one another; so that two out of these six combinations must be rejected: hence, all the possible combinations are 4: and as there is only one way in which the event mentioned can happen, or only one combination of two out of two, the probability is $\frac{1}{4}$. And it is plain that the probability is the same for either of the other combinations:—two tails, or head and tail. The probability of the specified combination failing is $1 - \frac{1}{4} = \frac{3}{4}$.

It is of no consequence whether the halfpence are thrown up simultaneously or one after the other; the intervention of time cannot influence the combination: instead of two halfpence, therefore, we may suppose the same halfpenny to be tossed up twice. In this case, however, the result of the first toss must be examined before the second is effected. If a sum of money, S, is to be received if both tosses give heads, and the proper consideration $\frac{1}{4}$ S forfeited if they do not, the

affair may be settled at the first toss, by *tail* turning up: in this case $\frac{1}{4}$ S, the price of the original expectation is lost, the expectation on the next toss being worth nothing: but if the first toss give head, then the expectation on the next toss is worth $\frac{1}{2}$ S. But if the result of the first toss be concealed from the parties interested, then, in virtue of their ignorance of it, the grounds of their expectation remain the same as at the outset; its value *to them* being still $\frac{1}{4}$ S.

3. What is the probability of throwing an ace in one throw with two dice?

Since, in throwing two dice each face of one may be combined with every face of the other, the total number of combinations is 36: of these, those only are favorable to the event mentioned in which an ace occurs. Now the ace on each die may be combined with every one of the six faces of the other in succession: the number of these combinations is therefore 12: but in these one of the combinations occurs twice; viz., the two aces: hence, the favorable cases amount to 11: therefore the probability of throwing an ace is $\frac{11}{36}$, and the probability of failure is $1 - \frac{11}{36} = \frac{25}{36}$.

If the condition be that one ace and no more is to be thrown, then the combination of the two aces must be rejected, and the probability of succeeding will be $\frac{10}{36}$; that of failing $\frac{26}{36}$. In this case each ace is combined with only *five* sides of the other in succession, all the favorable cases being 10.

The results would be the same if one die only be used and thrown twice.

4. What is the probability of throwing an ace *once* and not oftener in four throws of a die, or in a single throw with four dice?

The total number of combinations is found thus:

Each face of one die may combine successively with every face of a second, thus giving 36 combinations of two; each of these may combine with the several faces of a third die, giving $36 \times 6 = 216$ combinations of three; and lastly, each of these, combining with the faces of the fourth die, give $216 \times 6 = 1296$ for the total number of combinations.* In the favorable cases each ace unites in succession

* This number is also found at once by prob. v, page 351, to be $6 \times 6 \times 6$, since n, n', n'' are each 6.

with the combinations furnished by the other die regarding *five* faces of each only: these combinations, found as above, are $5^3 = 125$; therefore for the four aces in succession, the favorable combinations are 4×125. Hence the probability is $\frac{4 \times 125}{1296} = \frac{125}{324}$.

(10.) The following is another general principle of importance.

II. *If it is possible for an event to happen in n different ways, and if the probability of its happening in one of the ways be p_1, the probability of happening in a second way p_2, in a third way p_3, and so on to p_n: then the probability that it will happen in one or other of the n specified ways will be the sum of the individual probabilities, $p_1 + p_2 + p_3 + \dots + p_n$.*

Suppose a sum of money S receivable upon the happening of the event: then the value of all the n individual expectations is $p_1 S + p_2 S + p_3 S \dots + p_n S = (p_1 + p_2 + p_3 + \dots + p_n) S$: but it cannot possibly happen in more than one of the n ways: hence, that it *will* so happen the probability is $p_1 + p_2 + p_3 \dots + p_n$, this being the quantity which multiplied by S gives the value of the expectation of obtaining S.

5. As an application of this principle let us take example 1 before given.

The probability of drawing *either* two white balls or two black balls is $\frac{1}{8} + \frac{3}{8} = \frac{1}{2}$. Hence the probability of *not* drawing either, that is, of drawing one of each, is $1 - \frac{1}{2} = \frac{1}{2}$; so that the probability of drawing a ball of each kind is equal to that of drawing two of the same kind.

Or thus: we have seen that the probability of drawing one black ball *at least* is $\frac{7}{8}$; this, therefore, must be made up of the probabilities of drawing one black and one white, and of drawing two black: but the probability of drawing two black balls is $\frac{3}{8}$; hence $\frac{7}{8} - \frac{3}{8} = \frac{1}{2}$ the probability of drawing one black *only* and one white. In the same way the probability of drawing one white ball *at least* is $\frac{5}{8}$; this consists of the probability of drawing one white and one black, and of that of drawing two white balls: but the probability of drawing two white balls is $\frac{1}{8}$; therefore $\frac{5}{8} - \frac{1}{8} = \frac{1}{2}$ the probability that one will be white and the other black.

(11.) Want of space precludes our enlarging on this principle, which however will come into operation in connexion with that which follows, viz.:

III. *If any number of independent events can happen in conjunction, the probability of their so happening is equal to the product of the individual probabilities of their happening separately.*

Let the individual probabilities be $\frac{a}{n}$, $\frac{a'}{n'}$, $\frac{a''}{n''}$, &c. where a, a', a'', &c. are the numbers expressing the cases favorable to the respective events; and n, n', n'', &c. *all* the possible cases: then the cases favorable to the concurrence of the events will be expressed by all the combinations of 1 out of a, with 1 out of a', with 1 out of a'', &c.; and all the possible cases by all the combinations of 1 out of n, with 1 out of n', with 1 out n'', &c. By Prop. v, page 351, these several combinations are $aa'a''$, and $nn'n''$, &c.; hence the probability of the compound event is $\frac{aa'a'', \text{\&c.}}{nn'n'' \text{\&c.}}$

6. Required the probability of throwing an ace with a single die, in two trials.

This example has been considered before (page 355): we repeat it here as furnishing one of the simplest applications of the preceding principle.

The probability of succeeding the first time is $\frac{1}{6}$; of failing $\frac{5}{6}$. And the same as respects the second time. Therefore, by the above principle, the probability of failing both times is $\frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$: and consequently the probability of not failing both times, that is, of succeeding once, at least, is $1 - \frac{25}{36} = \frac{11}{36}$.

Or thus: The probability of succeeding the first time is $\frac{1}{6}$: the probabilities of failing the first time and succeeding the second, are respectively $\frac{5}{6}$ and $\frac{1}{6}$; therefore (Principle III), the probability of the compound event is $\frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$. Hence (II) the probability that either one or other of the throws will succeed is $\frac{1}{6} + \frac{5}{36} = \frac{11}{36}$.

7. What is the probability of throwing an ace once and not oftener, in four throws? (page 355.)

The probability of throwing an ace in any specified throw is $\frac{1}{6}$, and that of failing in all the other three throws (III) is $\left(\frac{5}{6}\right)^3$. Hence the

probability of the compound event (III) is $\frac{5^3}{6^4}$. But this compound event may happen in either of four different ways, as either throw may produce ace; therefore (II) the total probability is $4\frac{5^3}{6^4}=\frac{125}{324}$.

The probability that exactly *two* aces may turn up in four throws, may be found in a similar manner:—the probability that any two specified throws may each be ace is $\left(\frac{1}{6}\right)^2$, and that the other two throws may each fail to produce ace is $\left(\frac{5}{6}\right)^2$: therefore the probability of the compound event is $\frac{5^2}{6^4}$. But in four dice there are six combinations of two; $\therefore 6\frac{5^2}{6^4}=\frac{25}{216}$ is the probability that some two throws will each produce ace, and the other two fail.

The probability of throwing three aces exactly, that is, of three throws turning up ace and the remaining throw a different number—since this different number may turn up at either of the four throws—is $\left(\frac{1}{6}\right)^3\times\frac{5}{6}\times 4=\frac{20}{1296}=\frac{5}{324}$. And the probability of throwing four aces is $\left(\frac{1}{6}\right)^4=\frac{1}{1296}$: Consequently (II) the probability of either one or the other occurring, that is, of throwing three aces, at least, is $\frac{20}{1296}+\frac{1}{1296}=\frac{21}{1296}=\frac{7}{432}$.

(12.) A general formula for all questions of this kind may be investigated, by aid of the last two principles, as follows:

If the probability of the happening of an event in a single trial be p, and the probability of its failing, or the value of $1-p$ be q; to find the probability of happening once, twice, thrice, &c., exactly in n trials.

The probability of any specified trial succeeding and the remaining $n-1$ trials failing, is pq^{n-1}, and as there are n ways in which this may happen—that is, as either of the n trials may be specified—the probability that one or other will succeed and the rest fail is npq^{n-1}.

The probability of the event happening twice in two specified trials, and failing in the remaining $n-2$ trials, is p^2q^{n-2}: and as all the combinations of two out of n things are $\frac{n(n-1)}{1.2}$ in number, either of which may be the two specified, the probability that one or other of

the compound events will happen and the rest fail, is $\frac{n(n-1)}{1.2}$ p^2q^{n-2}.

The probability of the event happening three times in three specified trials, and failing in the $n-3$ remaining trials, is p^3q^{n-3}: the number of combinations of three out of n things is $\frac{n(n-1)(n-2)}{1.2.3}$; and either of these may be the combination specified: hence the probability that one or other will succeed and the rest fail is $\frac{n(n-1)(n-2)}{1.2.3}$ p^3q^{n-3}. (Principle II.)

And generally the probability that one or other of a set of h trials out of the n will succeed, and the remaining $n-h$ trials fail, is

$$\frac{n(n-1)(n-2)\ldots\ldots n-\overline{h-1}}{1.2.3\ldots\ldots h}p^hq^{n-h} \qquad [C].$$

In other words, *the probability of an event happening exactly h times in n trials is expressed by the $(n-h+1)$th term of the development of the binomial $(p+q)^n$ where p, q are the respective probabilities of happening and failing in a single trial.*

The probability that the event will happen *at least* h times in n trials is the same as the probability that it will happen *either* n times, or $n-1$ times, or $n-2$ times, or &c., down to h times; which by (II) is the sum of the probabilities of these occurrences individually.

Hence *the probability of happening at least h times in n trials is expressed by the sum of the first $n-h+1$ terms of the development of the binomial $(p+q)^n$.*

In applying the formula [C] we may of course avail ourselves of the property of the coefficients of the binomial development mentioned at (4): thus, for the probability of throwing an ace once, and not oftener, in four throws, in which case

$$p=\frac{1}{6},\ q=\frac{5}{6},\ h=1,\text{ and } n=4,$$

where $n-h>h$, the coefficient is most readily computed by the form in which it appears above: the required probability being

$$\frac{4}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3=\frac{125}{324}$$

But in computing the probability of throwing an ace three times, and not oftener, in which case

$$p = \frac{1}{6},\ q = \frac{5}{6},\ h = 3, \text{ and } n = 4,$$

where $n - h < h$, it will be better to put $n - h$ for h in the *coefficient*, thus changing [C] into

$$\frac{n(n-1)(n-2)\ldots\ldots(h+1)}{1.2.3\ldots(n-h)} p^h q^{n-h} \qquad [C']$$

and this, in the present instance, is

$$\frac{4}{1}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right) = \frac{5}{324}.$$

8. Five halfpence are tossed into the air: required the probabilities of the different combinations?

$$(p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5.$$

The probability p, of head turning up in a single trial is $\frac{1}{2}$: the probability q, of tail turning up is also $\frac{1}{2}$. Hence the probabilities of the several combinations are as follow:

The probability of all heads or all tails $= \frac{1}{2^5}$

4 heads and 1 tail, or 4 tails and 1 head $= \frac{5}{2^5}$.

3 heads and 2 tails, or 3 tails and 2 heads $= \frac{10}{2^5}$.

Therefore (II) the probability that they will be *either* all heads or all tails is $\frac{1}{2^5} + \frac{1}{2^5} = \frac{1}{2^4}$; that they will be *either* four heads or four tails exactly, $\frac{10}{2^5} = \frac{5}{2^4}$; and that they will be *either* three heads or three tails exactly, $\frac{20}{2^5} = \frac{5}{2^3}$. And the probability that one at least will be head is $\frac{1}{2^5} + \frac{5}{2^4} + \frac{5}{2^3} = \frac{31}{32}$; and the same of course is the probability that one at least will be tail.

(13.) It is worthy of remark that neither of the general forms [C], [C′] need be borne in the memory. All that is requisite is that we write down the probability for the proposed event succeeding h times in h *specified trials* out of all the n trials, and failing in the others; the expression for which, as we have already seen (p. 358), is $p^h q^{n-h}$: this done, we are to observe which of the two exponents here exhibited is the smaller; and to write, for the denominator of the coefficient to be prefixed to the expression, the factors 1 . 2 . 3, &c. up to this smaller exponent; and for the corresponding numerator as many of the factors $n(n-1)(n-2)$ &c. Thus, in the preceding example, for head to turn up in each of three specified trials out of five, and to fail to turn up in the remaining two, the probability is p^3q^2; therefore that three heads and two tails may turn up in five trials—without regard to any specified order—the probability is $\frac{5\,.\,4}{1\,.\,2}p^3q^2$, the exponent of q suggesting the denominator of the coefficient, and thence the numerator. In like manner, in throwing a die four times, $\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2$ is the probability that ace will turn up in each of two specified trials, and fail to turn up in each of the other two: therefore that some two throws—without particularizing which—will each bring up ace, and the other two a different point, the probability is $\frac{4\,.\,2}{1\,.\,2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2$.

(14.) It is of importance to observe further, that the probability of an event happening h times *at least*, may often be determined more readily than by summing up the first of $n-h+1$ terms of the development of $(p+q)^n$, as directed above; as will appear from the following considerations: Since $p+q=1$, therefore $(p+q)^n=1$; that is, the sum of all the terms of the development is unity: moreover the number of these terms is $n+1$. Consequently, when the above-mentioned $n-h+1$ terms are taken away, there must be h terms left:—the last h terms. If these, therefore, be taken from the sum of all, that is from unity, the remainder will be the sum of the first $n-h+1$ terms. But the last h terms of $(p+q)^n$ are the same as the first h terms of $(q+p)^n$. Hence the second proposition at page 359 may be otherwise expressed, as follows:

The probability of happening at least h times in n trials, is expressed

by unity minus the sum of the first h terms of the development of the binomial $(q+p)^n$, *where p and q are the respective probabilities of happening and failing in a single trial.*

Thus: in the preceding example, the probability of head turning up at least *once* in the five trials, is $1-\left(\frac{1}{2}\right)^5=\frac{31}{32}$; and this is a speedier method of determining the probability than that employed above. In like manner, if the following example were proposed, viz.

9. To determine in a single throw with nine dice, or in nine throws with a single die, what the probability is of throwing three aces at least? We should have $p=\frac{1}{6}$, $q=\frac{5}{6}$ and $h=3$. And the first three terms of $\left(\frac{5}{6}+\frac{1}{6}\right)^9$ being $\left(\frac{5}{6}\right)^9+9\left(\frac{5}{6}\right)^8\left(\frac{5}{6}\right)+36\left(\frac{5}{6}\right)^7\left(\frac{1}{6}\right)^2$, the probability sought would be unit minus the sum of these;—a result much more easily computed than the sum of the first seven terms of $\left(\frac{1}{6}+\frac{5}{6}\right)^9$.

(15.) It must be specially observed, in questions like those discussed above, that the events are always understood to be quite independent of one another, all the circumstances being the same for each trial: when such is not the case, that is, when the trials themselves modify the circumstances connected with those that are to follow, account must of course be taken of these modifications. A single example will sufficiently illustrate this:

10. A lottery consists of 100 tickets, of which four are prizes and the rest blanks: required the probability that in the first three tickets drawn there shall be at least one prize?

The probability of drawing a blank the first time is $\frac{96}{100}$. If a blank were certainly drawn, only 95 blanks would be left, and therefore the probability of the second drawing being a blank would be $\frac{95}{99}$: the compound probability—that is, that the first and second drawings are both blanks—is therefore $\frac{96}{100}\times\frac{95}{99}$. Were this compound event certain, the probability of the third drawing being a blank would, in like manner, be $\frac{94}{98}$. Hence the probability of all these three events is $\frac{96}{100}\times\frac{95}{99}\times\frac{94}{98}=\frac{7144}{8085}$: and consequently the probability that all

the drawings will not be blanks, that is that one, at least, will be a prize is $1 - \frac{7144}{8085} = \frac{941}{8085}$.

(16.) The examples proposed in the preceding articles to illustrate the first principles of the doctrine of Probabilities, have in themselves no useful connexion either with the objects of practical science or with the concerns of our moral or social existence: they rather indeed appear to bear upon a subject whose tendency is immoral, and influence on society pernicious;—the subject of *Gaming*. But it is so much more easy to convey the first elementary notions of this science by a reference to a throw of a die, or the toss of a coin, than by an appeal to the less palpable circumstances connected with the fluctuations of human life, human testimony, or other such contingencies, that it would be unwise to abandon this method of exposition merely because of its apparent association with chances at play, or with the disreputable craft of the gambler.

After the discussion of problems such as the foregoing, it is easy to see how they may be assimilated to inquiries of a more important character: the throws of a die, for example, may be compared with the statements of a witness whose veracity is so low that he may fairly be expected to tell five falsehoods out of every six statements he makes. If two such witnesses depone to any event—supposing no collusion—the probability that they will concur in the truth is only $\frac{1}{36}$, that they will both tell falsehood $\frac{25}{36}$: that three such witnesses will unite in telling the truth the probability is only $\frac{1}{216}$, and that they will all tell falsehood the probability is $\frac{125}{216}$; that is, the odds against the truth of their united testimony is 1 to 125. The greater the number, therefore, of *such* witnesses, the greater would be the improbability of the event to which they testify, so far as that event depends for credibility upon the value of their evidence in its favour.

Let it be remembered, however, that the event deponed to is here supposed to be given; that is, that the testimony is offered in reference to a *specified fact*. If such be not the case, but that, on the contrary, the witnesses come voluntarily forward to affirm the occurrence of some remarkable event—the performance of a miracle for instance—and that, without collusion, they all testify to the *same* miracle, the

simple fact of this concurrence in their testimony, if the witnesses be at all numerous, will give a degree of probability to the truth of their statement which is altogether irresistible, however abandoned the characters of the witnesses may be.

Suppose, for instance, that a person testifies to the fact that he saw a dead man raised to life: and that ten other persons with every disposition to deceive, but without collusion, testify to the same thing. Now, supposing that these ten persons were limited within the very narrow range of only ten fabrications equally suitable to their purposes of deception:—the probability that they would all fix upon the particular miracle mentioned is $\frac{1}{(10)^{10}} = \frac{1}{10000000000}$. If instead of ten persons there were twelve, the probability would be $\frac{1}{1000000000000}$, that is, the odds against the occurrence of this supposed uniformity of testimony is within a unit of a *million millions* to *one*, on the supposition that the pretended miracle is a fabrication. This conclusion is incontrovertible: and, like the other deductions from the theory of probabilities, will be found in strict accordance with the suggestions of common sense. If a notorious liar meet us in the street, and inform us that a murder has just been committed at a certain place, we give little or no credence to his statement, but if another person equally void of integrity and who, we know, cannot have communicated with the former, tell us the same story, we cannot resist a strong impression in favour of its truth,—we place no value upon the veracity of our informants, but a very high one upon the improbability of two fabricators of falsehoods independently inventing the same precise falsehood at the same time:* if a third person of like character repeat to us the information, we place as much reliance on the truth of the statement as if it had been made by a person of unquestionable veracity. The evidence of bad characters is sometimes rejected in courts of justice, on the ground that they are not to be believed on oath: yet if several of these bear independent testimony to the same event out of a number of other events equally likely to suggest themselves, or to

* As Mr. De Morgan justly remarks, in reference to cases of this kind, "Whether they be good or bad witnesses individually can only affect the rational probability of their assertion, in the manner in which it affects our disposition to suspect combination." (*Encyclopædia Metropolitana*, Art. "Probabilities.")

have occurred, we see that a high degree of probability would attach to the statement.

The foregoing conclusion, respecting the concurrent testimony of thirteen witnesses, has an obvious bearing upon Hume's celebrated argument on miracles, in which he attempts to prove that no amount of human testimony in favour of a miracle can ever render its occurrence so highly probable, as the uniform experience of mankind against it renders it improbable.

This uniform experience of the whole human race since the creation, in favour of the non-occurrence of a miracle, on the highest computation, is about two hundred thousand millions to one.* We have seen above, that if only thirteen individuals bear independent testimony to the fact—even supposing that they have only ten other events to choose from—the probability that that fact is not a fabrication is a million millions to one,—a probability far surpassing the former.

The point mainly insisted upon in Hume's argument is the fallibility of human testimony, which, either from want of integrity in the witnesses, or want of sagacity to detect fraud, has often deceived us, while "the laws of nature" never have. The considerations offered above are altogether independent of the character of the witnesses, and remain unaffected by every hypothesis as to their honesty or their intellect.

(17.) Let us now briefly inquire into the probability in support of a *specified* miracle, in reference to which alone the testimony of persons affirming themselves to be eye-witnesses is to be received, deriving our conclusions from the combined considerations of the probability of the testimony, and the inherent improbability of the miracle. Here all depends upon the number and integrity of the witnesses: let the probability of falsehood be p for each, their number being n; collusion is of course, as before, out of the question: indeed to suppose it would destroy the hypothesis of any respectable degree of integrity in the witnesses, an hypothesis here assumed.

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* Assuming the origin of the human race to have been about 6000 years ago, and taking 30 years as the duration of a generation, we have $\frac{6000}{30} = 200$ generations: and allowing that the average population of the earth has been a thousand millions, we find that there have been born and have died, since the creation, 200,000,000,000 individuals. (BABBAGE's *Bridgewater Treatise*, p. 138.)

Let m be the number of persons who have died without resurrection, the next person will be the $(m+1)$th, so that out of these $m+1$ persons the event cannot possibly happen but once, and all the ways of happening and failing are $m+2$: hence the probability of happening is $\frac{1}{m+2}$, and therefore the probability of not happening is $\frac{m+1}{m+2}$. Also the probability that the witnesses agree in speaking the truth is $(1-p)^n$, and that they concur in falsehood the probability is p^n. For the probability of the happening of the event—as inferred from the whole of the evidence—we must divide the result of the hypotheses favorable to the happening by the sum of the results of both hypotheses, favorable and unfavorable: we thus have for the probability of the witnesses speaking truth, and the event occurring,

$$\frac{(1-p)^n \frac{1}{m+2}}{(1-p)^n \frac{1}{m+2} + p^n \frac{m+1}{m+2}} = \frac{(1-p)^n}{(1-p)^n + p^n(m+1)};$$

and for the probability of the contrary, or the improbability of the event,

$$\frac{p^n(m+1)}{(1-p)^n + p^n(m+1)}.*$$

In order, therefore, that the inherent improbability of the event, when combined with the neutralizing force of the testimony, may render that event, all circumstances considered, more probable than improbable, we must have

$$(1-p)^n > p^n(m+1),$$

$$\text{or } \left(\frac{1}{p}-1\right)^n > m+1;$$

$$\therefore n \log\left(\frac{1}{p}-1\right) > \log(m+1) \therefore n > \frac{\log(m+1)}{\log\left(\frac{1}{p}-1\right)}.$$

* These are the expressions used by Mr. Babbage, from whom the line of argument in the present article is taken. But in what follows from this point, the reasoning and ultimate conclusion are different. Mr. Babbage's work abounds in highly interesting and original views on subjects of moral and philosophical importance, and is well worthy of attentive perusal.

Let $m + 1$, or the number of persons since the creation be a million millions: let also each witness tell one falsehood for every nine truths; that is, let $\frac{1}{p} = 10$, then

$$n > \frac{\log 10^{12}}{\log 9} > \frac{12}{\cdot 954} > 12.$$

Consequently, any number of witnesses greater than 12, whose veracity is such that they tell only one falsehood to nine truths, testifying to the truth of an event, the improbability of whose occurrence is a million millions to one, renders that event more likely to have happened than not.

From these investigations it follows, that if Hume had been at all acquainted with the true theory of probabilities, about which he has reasoned so largely and so loosely, he would never have propounded the argument against the possibility of miracles, which has procured for him such unenviable celebrity.

LIFE ANNUITIES.

(18.) We have devoted so much space to the subjects discussed in the preceding articles of this Appendix, as already to have exceeded the limits within which the present work was to have been comprised: we are precluded, therefore, from giving any more than a very few brief illustrations of the application of the theory of Probabilities to Annuities on Lives.

(19.) The bases of all calculations of Annuities on Lives are *Tables of Mortality*: that is to say, the recorded results of observations made during a series of years upon a large number of persons, in reference to the rate at which that number diminishes, year after year, by deaths, till all become extinct. Such tables have been formed by Dr. Price for the town of Northampton; by Mr. Milne for Carlisle; by Dr. Haygarth for Chester; and by Messrs. G. Davies and Finlaison, from the experience derived from the Insurance Offices in London, with which those gentlemen are connected. A specimen of the first two of these tables is here given.

Tables of Mortality at Northampton and Carlisle.

Age.	Northampton. Living.	Carlisle. Living.	Age.	Northampton. Living.	Carlisle. Living.	Age.	Northampton. Living.	Carlisle. Living.	Age.	Northampton. Living.	Carlisle. Living.
10	5675	6460	26	4685	5836	42	3482	4940	58	2202	3842
11	5623	6431	27	4610	5793	43	3404	4869	59	2120	3749
12	5573	6400	28	4535	5748	44	3326	4798	60	2038	3643
13	5523	6368	29	4460	5698	45	3248	4727	61	1956	3521
14	5473	6335	30	4385	5642	46	3170	4657	62	1874	3395
15	5423	6300	31	4310	5585	47	3092	4588	63	1793	3268
16	5373	6261	32	4235	5528	48	3014	4521	64	1712	3143
17	5320	6219	33	4160	5472	49	2936	4458	65	1632	3018
18	5262	6176	34	4085	5417	50	2857	4397	66	1552	2894
19	5199	6133	35	4010	5362	51	2776	4338	67	1472	2771
20	5132	6090	36	3935	5307	52	2694	4276	68	1392	2648
21	5060	6047	37	3860	5251	53	2612	4211	69	1312	2525
22	4985	6005	38	3785	5194	54	2530	4143	70	1232	2401
23	4910	5963	39	3710	5136	55	2448	4073	71	1152	2277
24	4835	5921	40	3635	5075	56	2366	4000	72	1072	2143
25	4760	5879	41	3559	5009	57	2284	3924	73	992	1997

(20.) By help of such tables as these, we may readily determine the probability of an individual of a given age surviving any proposed number of years.

For if A be the number of persons in the table living at the given age, and B the number living at the advanced age, then, since the individual has B chances in his favour out of A, the probability of his surviving the proposed term is $\frac{B}{A}$, and the probability of his not surviving is $\frac{A - B}{A}$ (page 353.)

The notation usually employed by writers on Annuities for the number of lives in the table aged a years, is l_a; so that the probability that a person aged a years will live to the age of b years, is $\frac{l_b}{l_a}$

Example 1. What is the probability that an individual aged 14 will attain the age of 21? (Carlisle Table.)

$$\frac{l_{21}}{l_{14}}=\frac{6047}{6335}; \text{ therefore probability of dying } = \frac{288}{6335}.$$

2. What is the probability that two individuals, one aged 14 and the other aged 23, will both live 7 years?

The probability that one event will happen is $\frac{l_{21}}{l_{14}}$, and that the other will happen, $\frac{l_{30}}{l_{23}}$: hence, that both will happen the probability is

$$\frac{l_{21}}{l_{14}}\times\frac{l_{30}}{l_{23}}=\frac{6047}{6335}\times\frac{5642}{5963}.$$

Unity, diminished by this product, is the probability that one, at least, will die; and the sum of the two fractions is the probability that one, at least, will live.

(21.) If a sum of money be dependent upon either of these contingencies, its prospective value is of course equal to that of the sum itself multiplied by the probability of receiving it. (Page 352.)

The *present* value of a sum receivable for certain, at the expiration of any number of years, has been determined at page 232: this present value, multiplied by the aforesaid probability, is the present value of the expectation. It is thus easy to find the sum to be paid immediately for the chance of receiving any proposed sum, provided an individual of a stated age survives any specified number of years.

3. What present sum must be paid to secure £850, provided a person now aged 14 attains the age of 21, interest being at 3 per cent.

If p be the present value of any sum certain receivable n years hence, computed as at page 232 (Cor. 3), then the present value of the same sum, contingent upon an individual aged m years living n years, is $p\frac{l_{m+n}}{l_m}$; in the present example, $p=$ £691·1278: therefore, by the Carlisle table

$$691{\cdot}1278\times\frac{6047}{6335}=659{\cdot}708=\text{£}659\ \ 14\ \ 2.$$

4. If £850 be receivable at the end of 7 years, provided two persons, one aged 14 and another aged 16, are both alive; required the present value, interest being at 3 per cent.?

$$p\frac{l_{21}}{l_{14}}\cdot\frac{l_{23}}{l_{16}}=691\cdot1278\times\frac{6047}{6335}\times\frac{5963}{6261}=£628\cdot308.$$

These examples sufficiently show how the value of a sum of money, otherwise certain, is to be estimated when its payment is dependent upon the continuance or failure of human life; and they suggest the manner of determining the correct sum to be paid down in order to secure a stipulated annual payment during the life of an individual of specified age, or during the joint existence of any number of individuals.

5. To find the present value of an annuity of £A to be paid during the life of an individual of any proposed age?

Let $\frac{1}{R}$ be the present value of £1, receivable at the end of a year (p. 232): then $\frac{A}{R^n}$ is the present value of £A receivable n years hence. Therefore, for a life aged m years, the present value of the first year's payment is $\frac{l_{m+1}}{l_m}\cdot\frac{A}{R}$; of the second year's payment $\frac{l_{m+2}}{l_m}\cdot\frac{A}{R^2}$; of the third $\frac{l_{m+3}}{l_m}\cdot\frac{A}{R^3}$; and so on to the extremity of life, that is, to the end of l_z of the tables. Consequently the present value of the annuity, that is, of all the payments, is

$$\frac{A}{l_m}\left\{\frac{l_{m+1}}{R}+\frac{l_{m+2}}{R^2}+\frac{l_{m+3}}{R^3}+\dots\frac{l_z}{R^{z-m}}.\right\}$$

The actual calculation of an annuity on a young life, by this formula, would be laborious: but the labour is superseded by a table, carefully constructed from the formula, for all values of m, and for different rates per cent. or values of R; A being assumed $=1l.$: this is called a 'Table of Annuities on Single Lives:' and as the annuity is uniformly supposed to be $1l.$, the number of pounds in the table opposite to l_m expresses the number of years' purchase that any annuity A on that life is worth: the rule, therefore, is to multiply the number of years' purchase found in the tables by the annuity: the product is the present value of the annuity.

6. To find the present value of an annuity of £A to be paid during the joint lives of two persons aged m and k years respectively?

As before, the present value of the first, second, &c. year's payment certain, is $\frac{A}{R}$, $\frac{A}{R^2}$, &c.: and these are to be multiplied by the respective probabilities of receiving it, viz. $\frac{l_{m+1}\,l_{k+1}}{l_m l_k}$, $\frac{l_{m+2}l_{k+2}}{l_m l_k}$, &c. Consequently, the present value of all the payments, that is of the whole annuity, is

$$\frac{A}{l_m l_k}\left\{\frac{l_{m+1}l_{k+1}}{R}+\frac{l_{m+2}l_{k+2}}{R^2}+\frac{l_{m+3}l_{k+3}}{R^3}+\ldots\right\},$$

the series within the brackets to be continued, as before, to the end of the tables. The results of this formula, like those of the preceding, are computed and arranged in tables of annuities on joint lives.

7. To find the present value of an annuity on two joint lives, to expire with the last of those lives.

The respective probabilities that the lives m, k, will drop the first year, are $1-\frac{l_{m+1}}{l_m}$, $1-\frac{l_{k+1}}{l_k}$; that they will drop the second year, $1-\frac{l_{m+2}}{l_m}$, $1-\frac{l_{k+2}}{l_k}$; and so on. Hence the probability that both lives will become extinct the first year is $\frac{l_m-l_{m+1}}{l_m}\cdot\frac{l_k-l_{k+1}}{l_k}$, the second year $\frac{l_m-l_{m+2}}{l_m}\cdot\frac{l_k-l_{k+2}}{l_k}$, &c. Therefore the probability that both will *not* become extinct the first year, the second year, &c., is

$$1-\frac{l_m-l_{m+1}}{l_m}\cdot\frac{l_k-l_{k+1}}{l_k},\ 1-\frac{l_m-l_{m+2}}{l_m}\cdot\frac{l_k-l_{k+2}}{l_k},\ \&c.$$

If we take any one of these expressions, the first for instance, and reduce to a common denominator, we find it become

$$\frac{l_k l_{m+1}+l_m l_{k+1}-l_{m+1}l_{k+1}}{l_m l_k}=$$

$$\frac{l_{m+1}}{l_m}+\frac{l_{k+1}}{l_k}-\frac{l_{m+1}l_{k+1}}{l_m l_k}$$

Consequently, by the formulas in the last two propositions, we infer that the present value of an annuity on two joint lives, to continue till

the last of them expires, is found by subtracting from the sum of the values of the annuity on each single life, the value on the two joint lives: hence the value in this case is easily found from the tables before referred to.

(22.) To find the present value of £A, to be paid at the end of the year in which a person m years of age shall die.

The sum £A, thus receivable at the end of the year in which a life becomes extinct, is called an *Assurance* on that life: and we are here required to find the present value of such an assurance. The method of proceeding is analogous to that in example 5. The present value of the expectation of receiving the sum £A at the end of any year, is obviously the present value of that sum certain at that time, multiplied by the probability of dying in the proposed year. Hence, putting l_m, as before, for the number living in the mth year of their age, and d_m for the number who die in that year, we have, for the present value of the assurance—the sum of the values of all the expectations—

$$\frac{\text{A}}{l_m}\left\{\frac{d_{m+1}}{\text{R}}+\frac{d_{m+2}}{\text{R}^2}+\frac{d_{m+3}}{\text{R}^3}+\dots\right\}.$$

There is of course a certain sum of money x, which if paid annually will be equivalent to the present payment of the sum expressed by the above formula: this sum x is called the annual *premium;* it differs, however, from an annuity in this:—that the first payment is made at the *commencement* of the year—at the time of effecting the insurance—and thus there is one payment more than in the case of an annuity. We may therefore consider the above formula to express the present value of an annuity x on the proposed life, increased by x; and since the present value of an annuity of x pounds is x times the present value of an annuity of £1, it follows that

Present val. of annuity of £1, + 1 : £1 :: Present val. of assurance : x,

$$\therefore x = \frac{\text{Present val. of assurance}}{\text{Present val. of annuity of £1, + £1}} = \text{annual premium};$$

so that the *annual premium* for insuring £A at death is equal to the present value of the assurance computed by the formula—and which is called the *single premium*—divided by the present value of an annuity of £1 for the proposed life, increased by £1. But, as in the former problems so here, the labour of executing calculations of this kind is

very considerably reduced by means of tables constructed for the purpose: of these, the most copious and useful collection is that published in the 'Library for Useful Knowledge,' in connexion with the 'Treatise on Annuities and Assurances,' by Mr. DAVID JONES; to which the student is referred for ample information on these subjects. And for further details on the other matters discussed in this Appendix, he may consult the 'Treatise on Probability,' by Sir J. W. LUBBOCK, in the 'Library of Useful Knowledge;' the Essay on the same subject, by Mr. DE MORGAN, in 'LARDNER'S Cyclopædia;' the more extensive treatise, by the same author, in the 'Encyclopædia Metropolitana;' and Mr. GALLOWAY'S treatise, as reprinted from the 'Encyclopædia Britannica.'

NOTE A, (*page* 237.)

The amount of £1 for one year, increasing at compound interest, due at every xth part of a year, is, as in the text,

$$A = a\left(1+\frac{r}{x}\right)^x,$$

which, as we have already shown, is calculable, even when x is infinitely great. It may be inquired, however, whether A in these circumstances be the greatest possible or not. This may be ascertained as follows.

By the binomial theorem,

$$A = a\left(1+\frac{r}{x}\right)^x = a\left\{1+r+\frac{x(x-1)}{2}\cdot\frac{r^2}{x^2}+\frac{x(x-1)(x-2)}{2\,.\,3}\cdot\frac{r^3}{x^3} + \&c.\right\};$$

and since, for any finite value of x, $x(x-1)$ is less than x^2; $x(x-1)(x-2)$ less than x^3, &c., it follows that

$$a\left(1+\frac{r}{x}\right)^x < a\left\{1+r+\frac{r^2}{2}+\frac{r^3}{2\,.\,3}+\frac{r^4}{2\,.\,3\,.\,4}+\&c.\right\}.$$

But when x is infinite, the proposed series becomes

$$a\left(1+\frac{r}{\infty}\right)^\infty = a\left\{1+r+\frac{r^2}{2}+\frac{r^3}{2\,.\,3}+\frac{r^4}{2\,.\,3\,.\,4}+\&c.\right\}$$

*

Hence, when $x = \infty$, A becomes a maximum, and equal to this last series, which is the development of $a \times e^r$ (p. 222); therefore, as in the text, $\log A = \log a + r$.

The series just given is remarkable, showing that

$$\left(1 + \frac{1}{\infty}\right)^{\infty} = e = 2{\cdot}718281828 \text{ (page 223)};$$

By making $r = -1$, we have in like manner

$$\left(1 - \frac{1}{\infty}\right)^{\infty} = 1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \&c. = \frac{1}{e} \text{ (p. 223)}:$$

so that $1 \pm \frac{1}{\infty}$ must not be confounded with 1 when powers of it to ∞ are to be taken.

Note B, (*page* 277.)

In the article to which the present note refers, as also in the scholium at page 203, some remarks have been offered respecting the interpretation of the "&c." usually written after a finite number of the terms of an infinite series. This interpretation is a matter of some importance; and it is to the inattention with which it has hitherto been regarded that the perplexities and inconsistencies in the doctrine of diverging series—as delivered by modern analysts—are to be traced. We shall here briefly consider the meaning of this symbol in its connexion with geometrical series only.

By (76) the general expression for the sum of n terms of the series

$$a + ar + ar^2 + ar^3 + \&c. \;\ldots\ldots\; [A]$$

is

$$S = \frac{a}{1-r} - \frac{ar^n}{1-r} \;\ldots\ldots\; [A']$$

the &c. implying the endless progression of the terms beyond ar^3, agreeably to the law exhibited, and excluding everything in the form of supplement or correction.

Now it is customary to write the development of $\frac{a}{1-r}$ as follows

$$\frac{a}{1-r} = a + ar + ar^2 + ar^3 + \&c. \;\ldots\ldots\; [B],$$

and then to commit the mistake of confounding this with the series above; overlooking the fact that the &c. in the one, except under particular restrictions as to the value of r, is very different from that in the other; and it has hence been inferred that the equation [B], though

algebraically true, may nevertheless be arithmetically false; a statement, however, which is a contradiction in terms.

If we dispense with the &c. in the first of the above series, we may write it thus:

$$a + ar + ar^2 + ar^3 + \ldots\ldots + ar^{\infty}$$

the sum of which will be expressed by the formula [A′], by making n infinite; as that formula is perfectly general.

But this formula gives for $\frac{a}{1-r}$ the development

$$\frac{a}{1-r} = a + ar + ar^2 + ar^3 + \ldots\ldots + ar^{\infty} + \frac{ar^{\infty}}{1-r} \ldots\ldots [C];$$

showing that the &c. in [B] differs from that in [A] by a quantity which is infinitely great when r is not a proper fraction; except in the single case of $r = -1$. When r *is* a proper fraction, the two series comprehended under the &c. become identical by the evanescence of $\frac{ar^{\infty}}{1-r}$.

It thus appears that $\frac{a}{1-r}$ is not the fraction which generates the series [A], r being unrestricted; what this fraction really generates is exhibited in [C] above, an equation which is always true whatever arithmetical value we give to r: and to obtain the general expression for the sum of [A] we must connect to $\frac{a}{1-r}$ the correction $-\frac{ar^{\infty}}{1-r}$, a correction which is ambiguous, as to sign, when r is negative.

When r is > 1, the series, omitting this correction, is ∞; the correction itself is also ∞, and opposite in sign. The difference of these two infinites is the finite undeveloped expression: so that, as stated at page 205, the supposed development, when properly corrected, proves to be no development at all: and all the perplexities about diverging infinite series are explicable in the same way. When the terms of such series are alternately plus and minus, their sums are wrongly assigned in analysis: finite values being attributed to them when they are in reality infinite. But this we propose to show more fully elsewhere.

In the reasoning at page 276, we may seem to have employed $\frac{a}{1-r}$ as the *general* equivalent of [A]: but it will be seen that we restrict it in reality to the extreme of the cases, noticed above, of r equal to a proper fraction; this extreme case being $r = \pm 1$.

The particulars delivered in this Note furnish additional illustration of what is stated at page 117, respecting the developments of the two fractions

$$\frac{x^n - y^n}{x + y} \text{ and } \frac{x^n + y^n}{x + y},$$

which may be written

$$\frac{x^n}{x + y} - \frac{y^n}{x + y} \text{ and } \frac{x^n}{x + y} + \frac{y^n}{x + y}.$$

The first of these represents the *two* infinite series:

$$x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \&c.$$

$$- \left(\frac{y^n}{x} - \frac{y^{n+1}}{x^2} + \frac{y^{n+2}}{x^3} - \frac{y^{n+3}}{x^4} + \&c.\right)$$

of which series it is obvious that the second—disregarding signs—is the same as the first, commencing at the $n + 1$th term: the signs too will be the same in both, provided this $n + 1$th term be plus, like the leading term of the second series. Now the odd terms only of the first series are plus: hence if $n + 1$ be odd, that is, if n be even, the terms after the nth in the first series will be cancelled by the subtraction of the second series; so that the development will be limited to the n leading terms of the first series. If, on the contrary, $n + 1$ be even, that is if n be odd, the signs in the subtractive series will be the opposite of those of the like terms in the upper series: these terms, therefore, must all appear in the remainder, with the numeral coefficient 2 to each, and the development will be interminable. If the sign minus, prefixed to the second series, be changed into plus, the development of the second fraction will be exhibited; and the preceding conclusions will be reversed: that is if n be odd, the terms after the nth in the first series will be cancelled by the addition of the second series; and if n be even, these terms, with a coefficient 2 to each, must all appear in the sum.

THE END.

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